AGI

KEY

TO

THE NORMAL ELEMENTARY GEOMETRY AND TRIGONOMETRY.

BY

EDWARD BROOKS, A.M.,

PRINCIPAL OF PENNSYLVANIA STATE NORMAL SCHOOL, AND AUTHOR OF "NORMAL SERIES OF ARITHMETICS," "NORMAL ALGEBRA," "NORMAL GEOMETRY," "PHILOSOPHY OF ARITHMETIC," ETC.

PHILADELPHIA:
SOWER, POTTS & CO.,
530 MARKET ST. AND 523 MINOR ST.

WESTCOTT & THOMSON,
Stereotypers and Electrotypers, Phila.

SHERMAN & CO.,
Printers, Phila.

PREFACE.

THIS Key is designed especially for those who are studying Geometry and Trigonometry without a teacher. It will also be found of use to teachers who may not always have time to work out the problems or demonstrate the theorems, or who may wish to compare their own demonstrations and solutions with others. The shortest demonstrations have not always been given, but those which were supposed to be the simplest and the best. Some of the results of the problems will be found to vary slightly from those given in the book; this arises usually from the difference in the number of decimal places to which the calculation is extended. It is proper for me to add that this key has been prepared by a former pupil, a lady of superior mathematical ability, though nearly all of her work has passed under my own personal supervision, and is therefore endorsed as correct.

EDWARD BROOKS.

STATE NORMAL SCHOOL Jan. 10, 1875.

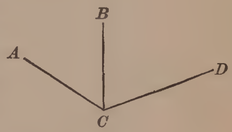
KEY

TO THE

NORMAL ELEMENTARY GEOMETRY.

BOOK I.

PRACTICAL EXAMPLES. (p. 43.)



2. If the lines DC and AC meet BC at the point C, making the angle ACB equal to 30°, and the angle BCD equal to 80°, then the angle ACD will equal $30° + 80°$, or 110°. But if we take BC and CD for the given lines, then BCD, the included angle, will equal $80° - 30°$, or 50°.

3. Since each angle of a rectangle is a right angle (Def. 28), each angle must contain 90°.

4. Since the angles of any triangle equal 180°, one of the three equal angles of an equilateral triangle equals $\frac{1}{3}$ of 180°, or 60°.

5. Since the three angles of a triangle equal 180°, and two of them equal $43° + 75° = 118°$, the third angle equals $180° - 118°$, or 62°.

6. The sum of two angles equals 90°, hence the other angle equals $180° - 90°$, or 90°. The triangle is right-angled isosceles.

7. The sum of the other two angles equals $180° - 60°$, or 120°; and since the angles are equal, one of them will be $\frac{1}{2}$ of 120°, or 60°. The three angles being equal, the triangle is equilateral.

8. If one of the equal angles is 30°, the other must also be 30°; hence, the third angle equals $180° - 60° = 120°$.

9. See Theorem XIX., Cor. 3. **10.** See Theorem XIX., Cor. 4.

11. Since A is twice and B three times C, the three angles $= C + 2\,C + 3\,C$, or $6\,C$, which equals 180°. Hence, $C = 30°$, $A = 60°$ and $B = 90$.

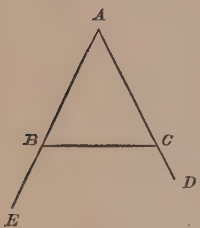
1*

12. A right-angled scalene triangle.

13. The three angles equal three times the sum of the equal angles; hence the sum of the equal angles equals $\frac{1}{3}$ of 180°, or 60°; each of the equal angles equals 30°, and the unequal angle equals 120°.

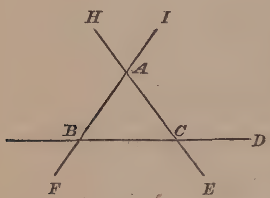
14. The sum of the exterior angles equals 4 right angles (Th. XIX.), or 360°; hence, each exterior angle of the octagon will equal $\frac{1}{8}$ of 360°, or 45°.

EXERCISES FOR ORIGINAL THOUGHT. (p. 44.)



1. Let ABC be an isosceles triangle, and the equal sides AB and AC be produced to E and D; then will the angle EBC equal the angle BCD.

For (Th. I.), $EBC+ABC$ equal two right angles, and also $ACB+BCD$ equal two right angles; therefore (Ax. 1) $EBC+ABC=ACB+BCD$. But $ABC=ACB$ (Th. X.); hence, $EBC=BCD$ (Ax. 3).

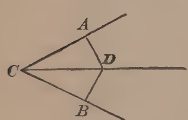


2. Let ABC be an equilateral triangle having the sides produced. Now, the angle $DCE=ACB$ (Th. II.), and $GBF=ABC$, and $HAI=BAC$. But ACB, ABC, and BAC are equal; hence (Ax. 1), DCE, GBF, and HAI are equal.

Again, $ACD+DCE=$two right angles, and $DCE+ECB=$ two right angles; hence (Ax. 3), $ACD=ECB$. In the same way ECB may be proved equal to CBF, CBF to GBA, etc.

3. Let ABC be any triangle; then, any side, as AB, is greater than the difference of the other two sides.

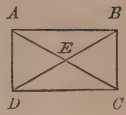
For (Ax. 10, Cor.), $AB+AC>BC$. Transposing, $AB>BC-AC$.



4. Let ACB be any angle bisected by the line CD; then will any point, as D, be equally distant from the sides of the angle.

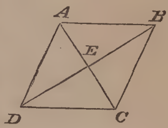
For, letting fall perpendiculars from D upon the sides of the angle (Th. XIV.), we have two triangles ACD and BCD, having the angles $ACD=DCB$ by construction, $CAD=CBD$, both being right angles, and therefore the remaining angles equal, and the side CD common; hence (Th. VII.), $AD=DB$.

5. Let $ABCD$ be any rectangle, and AC and BD its diagonals; then will AC and BD be equal.



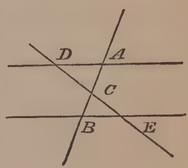
For, the triangles ABD and ABC have the side AB common, the side AD equal to BC, and the included angle BAD equal to the included angle ABC, both being right angles; hence (Th. VI.), the remaining sides AC and BD are equal.

6. Let $ABCD$ be a quadrilateral whose diagonals form right angles at E; then will the figure be a rhombus or square.



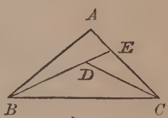
For, the two triangles AEB and BEC have the side AE equal to the side EC, being halves of the same diagonal, the side EB common and the angle AEB equal to the angle BEC, both being right angles; hence (Th. VI.), $AB = BC$. In the same way it may be proved that AD or DC is equal to AB or BC. Therefore the quadrilateral, having all its sides equal, is a rhombus or square.

7. Let AD and BE be two parallels, and AB a line joining them bisected in C; then will any other line, as DE, joining the parallels and passing through C, be bisected in that point.



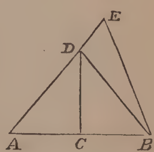
For, in the triangles ACD and BCE we have $AC = CB$ by construction, angle $ECB =$ angle ACD (Th. II.), and $DAC = CBE$ (Th. III., 2); hence, $CE = CD$ (Th. VII.).

8. Let ABC be a triangle, and from the point D let lines BD and DC be drawn; then will $DB + DC < AB + AC$.



For, producing BD to E, in the triangle AEB, $AB + AE > BE$ (Ax. 10, Cor.), and in triangle EDC, $ED + EC > DC$. Adding the inequalities, we have $AB + AE + ED + EC > BE + DC$. Taking ED from each member (Ax. 4), we have $AB + AE + EC > BD + DC$, or $AB + AC > BD + DC$.

9. Let AB be any line, having its middle point at C; then any point, as D, in the perpendicular CD will be equally distant from A and B, and any point E, out of the perpendicular, will be unequally distant from those points.



I. Joining the point D with A and B by lines, we have $AD = DB$ (Th. XIV., 2).

II. Joining the point E with A and B, the line EA will cut the perpendicular in some point as D. By the first part of the theorem DB

$=DA$. In the triangle DEB, $EB<DE+DB$, or substituting for DB its equal DA, $EB<DE+DA$, or $EB<EA$.

BOOK II.

PRACTICAL EXERCISES. (p. 52.)

1. Since the product of the means equals the product of the extremes (Th. I.), the 4th term $=\frac{14\times18}{12}=21$.

2. Since $3:12::5:20$, we have (Th. VI.) $3+12:12::20+5:20$, or $15:12::25:20$.

3. From Theorem III. we have $\sqrt[2]{12\times27}=18$; $\sqrt{m\times n}$.

4. $\frac{A}{B}=\frac{4}{9}$; multiplying by $\frac{3}{2}$, $\frac{3\,A}{2\,B}=\frac{12}{18}=\frac{2}{3}$.

5. $\frac{3\,A}{2\,B}=\frac{3}{4}$; dividing by $\frac{3}{2}$, $\frac{A}{B}=\frac{1}{2}$.

6. $M\times N=(A+B)(A-B)$; $\therefore$ $M:A+B::A-B:N$ (Th. II.).

7. $(C+D)\times A=(A+B)\times C$; $\therefore$ $A:A+B::C:C+D$ (Th. II.), and by division $A:B::C:D$.

THEOREMS FOR ORIGINAL THOUGHT. (p. 52.)

1. Suppose $a:b::c:d$;

then we have $\frac{a}{b}=\frac{c}{d}$;

multiplying by $\frac{m}{n}$, $\frac{am}{bn}=\frac{cm}{dn}$;

whence, $am:bn::cm:dn$.

2. Suppose $a:b::c:d$;

then by alternation, $a:c::b:d$;

multiplying the first couplet by $\frac{1}{m}$ and the second by $\frac{1}{n}$;

$$\frac{a}{m}:\frac{c}{m}::\frac{b}{n}:\frac{d}{n};$$

whence, $\frac{a}{m}:\frac{b}{n}::\frac{c}{m}:\frac{d}{n}$.

3. Suppose $a : b :: c : d;$
then by inversion, Theorem V., $b : a :: d : c,$
and by Theorem VI., $a+b : a :: c+d : c;$
and by inversion, $a : a+b :: c : c+d.$

4. Suppose $a : b :: c : d;$
by Theorem VI., $a+b : b :: c+d : d,$
and by Theorem IV., $a+b : c+d :: b : d;$
by Theorem VII., $a-b : b :: c-d : d,$
and by Theorem IV., $a-b : c-d :: b : d;$
whence, by Theorem VIII., $a+b : c+d :: a-b : c-d,$
and by Theorem IV., $a+b : a-b :: c+d : c-d.$

5. Suppose $a : b :: c : d,$ and $m : c :: n : d;$
then by Theorem IV., $m : n :: c : d;$
whence, by Theorem VIII., $a : b :: m : n.$

BOOK III.

PRACTICAL EXAMPLES. (p. 71.)

1. Perimeter $=4 \times 20=80$ in.; area $=20^2=400$ sq. in. Th. I.

2. Perimeter $=(18+24) \times 2=84$ in.; area $=18 \times 24=432$ sq. in.

3. Area $=16 \times 12=192$ sq. in. Th. II.

4. Area $=\dfrac{9 \times 1\frac{1}{2}}{2}=6\frac{3}{4}$ sq. feet. Th. III.

5. Area $=\dfrac{(40+60)}{2} \times 32=1600$ sq. rd. Th. IV.

6. The hypothenuse $=\sqrt{3^2+4^2}=\sqrt{25}=5$. Th. VI.

7. Ratio of bases $=\frac{8}{24}=\frac{1}{3}$; hence the other sides are $\frac{18}{3}=6$ and $\frac{21}{3}=7$.

8. $\sqrt{78^2-30^2}=72$ (Th. VI., Cor. 1). Area $=\dfrac{72 \times 30}{2}=1080$.

9. We shall have a right-angled triangle, the ladder being the hypothenuse and distance from the foot, the base; hence, the height is $\sqrt{65^2-25^2}=60$ ft.

10. The broken part of the pole will form the hypothenuse of a right-angled triangle, the upright part, the perpendicular, and the distance from the foot to where the top touches the ground, the base; hence, the upright part $=\sqrt{75^2-60^2}=45$; $45+75=120$ feet.

11. Let x = base of second triangle; then, by similar triangles, $1 : 4 :: 20^2 : x^2$; hence, $x = 40$ rods.

12. Let x = length of second lot, and y its width; then, by similar triangles, $1 : 9 :: 40^2 : x^2$; $\therefore x = 120$; also $1 : 9 :: 23^2 : y^2$; $\therefore y = 69$.

13. The length of the ladder will form the hypothenuse of a right-angled triangle, the distance it is drawn out at bottom will be the base, and the length of the ladder minus 7 feet, the perpendicular. Now, $91 - 7 = 84$, and $\sqrt{91^2 - 84^2} = 35$ feet.

14. The length of the ladder will be the hypothenuse of two triangles, the perpendiculars of which will be the heights of the two windows, and the sum of their bases will be the width of the street; hence, $\sqrt{130^2 - 78^2} = 104$; and $\sqrt{130^2 - 50^2} = 120$; 104 ft. + 120 ft = 224 ft.

15. Area $= 25 \times 16 = 400$; side of the square $= \sqrt{400} = 20$ yards.

16. Area of farm $= 50 \times 32 = 1600$ sq. rods; side of the square, $\sqrt{1600} = 40$ rods; perimeter of farm, $(50 + 32) \times 2 = 164$ rods; \$328 ÷ 164 = \$2, price of fencing 1 rod; $40 \times 4 = 160$ rods, perimeter of square; 160 × \$2 = \$320, price of fencing square; \$328 − \$320 = \$8, difference.

17. Area of one gable $= \dfrac{48 \times 10}{2} = 240$ sq. ft.; 2×240 sq. ft. $= 480$ sq. ft.

18. 40 acres = 6400 perches. This field may be divided into 2 equal squares, each containing 3200 perches. The side of 1 square $= \sqrt{3200} = 56.568$ rods. Length $= 2 \times 56.568 = 113.136$ rods. Or, let x = breadth and $2x$ = length of field. Then, $2x^2 = 6400$, whence $x = 56.568+$.

19. 60 acres = 9600 perches. This field may be divided into 3 equal squares, each containing 3200 perches, of which one side will be 56.568 rods. Length, $3 \times 56.568 = 169.704$ rods. Or, let x = breadth and $3x$ = length, then $3x^2 = 9600$, and $x = 56.568$.

20. $100 + 136 = 236$, difference of squares, and 2 = difference of sides. Then, $236 \div 2 = 118$, sum of sides; $\dfrac{118 - 2}{2} = 58$, side of lesser square; $58^2 + 100 = 3464$, *Ans.* Or, $236 - 4 = 232$, which equals 4 times the side of first square; hence, first square equals $232 \div 4 = 58$, and number of men equals $58^2 + 100 = 3464$.

21. $\frac{1}{10}$ of an acre = 16 square rods, and $\sqrt{16} = 4$ rods, the length of a side; $\frac{64}{64} - \frac{15}{64} = \frac{49}{64}$; hence, area of part enclosed is $\frac{49}{64} \times 16$ square rods, or $\frac{49}{4}$ square rods; and length of side equals $\sqrt{\frac{49}{4}}$, or $3\frac{1}{2}$ rods. Hence, twice the width is $4 - 3\frac{1}{2}$, or $\frac{1}{2}$ a rod, and the width is $\frac{1}{4}$ a rod, or 4 feet $1\frac{1}{2}$ inches.

22. The given triangle will be divided by the perpendicular into two right-angled triangles, of which the given sides will be the hypothenuses and the segments of the other side the bases; hence, one base $=\sqrt{13^2-12^2}=5$, and the other base $=\sqrt{15^2-12^2}=9$; and their sum, or base of the given triangle $=9+5$, or 14.

EXERCISES FOR ORIGINAL THOUGHT. (p. 73.)

1. Since the diagonal divides the square into two equal right-angled triangles, the squares will be to each other as the triangles; but two similar right-angled triangles are to each other as the squares of their hypothenuses, which are the diagonals of the squares.

2. Since the diagonal divides the parallelogram into two equal triangles, the triangles are to each other as the parallelograms; but the triangles are to each other as the squares of the diagonals, which are homologous sides.

3. This may be seen from the principle that the square of each is equal to the sum of the squares of equal sides.

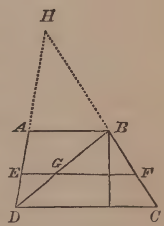
4. Taking the diagram in Book I., Theorem XVIII., the diagonal AC is greater than the diagonal BD.

For, in the triangles ADB and ABC, the two triangles have $AD=BC$, AB common, and the included angles unequal; hence (Bk. I., Th. VIII.), the side AC, which is opposite the greater angle, is greater than BD.

5. If a line be drawn from the vertex of a triangle bisecting the base, it will divide the triangle into two equal parts.

For, the two triangles thus formed will have the same altitude, and therefore are to each other as their bases (Th. III., Cor. 1), but the bases being equal the triangles must be equal.

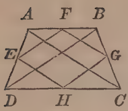
6. See Theorem VI., Cor. 3.



7. Let $ABCD$ be a trapezoid, and EF a line joining the middle points of the sides AD and BC; then will EF be parallel to AB and DC, and equal to half their sum.

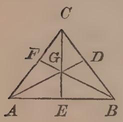
For, produce AD and BC till they meet in H. Then, since the line EF divides HD and HC proportionally, it is parallel to the base. (Th. IX., Cor. 2.)

Again, drawing the diagonal BD, the triangles ABD and GED are similar (Th. XIII.), and hence EG is one-half of AB. In the same manner GF may be shown to be one half of DC; hence, $EF=\frac{1}{2}(AB+DC)$.

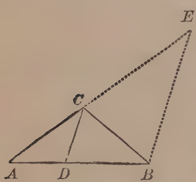


8. Draw the diagonals AC and BD. In the triangle ABD, the line EF bisects AB and AD, and therefore is parallel to BD (Th. IX., Cor. 2). For the same reason GH is parallel to BD; and hence EF and GH, having the same direction, are parallel to each other. In a similar way it may be shown that EH and FG are parallel; hence, $EFGH$ is a parallelogram.

9. In the triangle ABC the perpendiculars from the vertices bisect the sides (Bk. I., Th. X., Cor.). Then every point in the line CE is equally distant from the points A and B (Bk. I., Th. 9 for Original Thought). For the same reason every point in the line AD is equally distant from B and C; hence, their point of intersection G is equally distant from A, B, and C. But by the same theorem every point equally distant from A and C must be on the perpendicular BF; hence that line passes through the point G.



10. Let CD be the line bisecting the vertical angle of the triangle ABC. Draw EB parallel to CD, and produce AC till it meets EB in E. Then, since CD and BE are parallel, the angles ACD and CEB are equal (Bk. I., Th. III.), and also the angles DCB and CBE; hence, as ACD and DCB are equal by construction, CEB must be equal to CBE, and the sides CB and CE of the triangle BCE must be equal (Bk. I., Th. XI.).



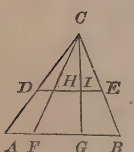
From the triangle AEB we have the proportion (Th. IX.),

$$AC : CE :: AD : DB;$$

or substituting CB for its equal CE,

$$AC : CB :: AD : DB.$$

11. Let ABC be any triangle, and DE a line parallel to the base, and CF and CG any lines drawn from the vertex to the base. The two triangles DCH and ACF being similar (Th. XIII.), we have the proportions



$$CH : CF :: DH : AF.$$

In like manner, from the triangles HCI and FCG we have

$$CH : CF :: HI : FG;$$

hence (Book II., Theorem VIII.), $DH : AF :: HI : FG.$

In the same manner we obtain the proportion, $HI : FG :: IE : GB;$

whence the continued proportion $DH : AF :: HI : FG :: IE : GB.$

12. See Theorem XV.

BOOK IV.

PRACTICAL EXERCISES. (p. 89.)

1. The circumference $=2\pi R$ (Th. XII., Cor.); substituting the value of R, we have $12\,\pi$.

2. The area of a circle equals $\pi\,R^2$ (Th. XIV.); substituting, we have $16\,\pi$.

3. The circumference equals $2\pi R$; hence, $R=\dfrac{\text{cir.}}{2\,\pi}=\dfrac{50.2656}{2\times 3.1416}=8$.

4. The area $=\pi\,R^2$; hence, $R=\sqrt{\dfrac{\text{area}}{\pi}}=12\frac{1}{2}$, diameter $=25$, and circumference $=2\pi R=78.54$.

5. Circumference $=2\,\pi\,R$; hence, $R=\dfrac{\text{cir.}}{2\,\pi}=28.647+$. The area $=$ circumference $\times\frac{1}{2}\,R=2578.23$ rods $=16$ A. 18.23 P.

6. An arc of $75°$ is $\frac{75}{360}$ or $\frac{5}{24}$ of a circumference. Circumference $=2\,\pi\,R=31.416$; $\frac{5}{24}$ of 31.416 feet $=6.545$ feet.

7. Circumference $=31.416$ feet $=376.992$ inches. An arc of 18 inches $=\dfrac{18}{376.992}$ of $360°=17°\ 11'\ 19''$.

8. Area of first $=10^2\times 3.1416=314.16$ square feet; area of second $=15^2\times 3.1416=706.86$ square feet; difference of areas $=706.86-314.16$ $=392.70$ square feet.

9. Let D represent the diameter of B's garden; then (Th. XI., Cor. 1), $D^2 : 18^2 :: 2\frac{7}{9} : 1$, or $D : 18 :: \frac{5}{3} : 1$; hence, $D=\frac{5}{3}\times 18=30$ rods.

10. The diameter of the circle becomes the diagonal of the inscribed square; hence, $5^2=$ sum of the squares of two sides of the square, and one side $=\sqrt{\frac{25}{2}}=3.535$ feet.

11. Radius of park $=\dfrac{160}{2\times 3.1416}$; radius of lake $=\dfrac{80}{2\times 3.1416}$; $\dfrac{160}{2\times 3.1416}-\dfrac{80}{2\times 3.1416}=\dfrac{80}{2\times 3.1416}=12.732$ rods.

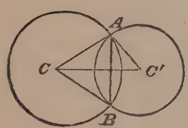
12. Diameter of circular garden $=\dfrac{120}{\pi}$; area $=\left(\dfrac{60}{\pi}\right)^2\times\pi=\dfrac{3600}{3.1416}$ $=1145.91$. Side of square garden $=\dfrac{120}{4}=30$; area of square $=900$. Difference $=1145.91-900=245.91$.

13. Radius of circular garden $= \sqrt{\frac{160}{3.1416}}$; circumference $= 160 \div \frac{1}{2}\sqrt{\frac{160}{3.1416}} = 2 \times \sqrt{3.1416 \times 160} = 44.84 -$ rods. Perimeter of square garden $= 4\sqrt{160} = 4 \times 12.649 = 50.596$ rods. Difference $= 50.596 - 44.84 = 5.756$ rods.

14. Radius $= \sqrt{\frac{314.16}{3.1416}} = 10$; hence, diameter $= 20$, and area of circumscribed square $= 20^2 = 400$. Difference $= 400 - 314.16 = 85.84$.

15. Radius $= \sqrt{\frac{640}{3.1416}}$; hence, side of inscribed square $= \sqrt{\frac{2 \times 640}{3.1416}}$; area of sq. $= \frac{2 \times 640}{3.1416} = 407.4357$; difference $= 640 - 407.4357 = 232.5643$ P; $= 1$ A. 1 R. 32.56 P.

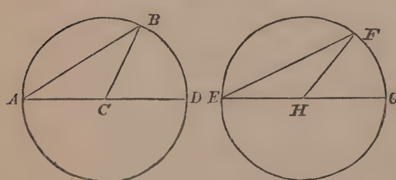
THEOREMS FOR ORIGINAL THOUGHT. (p. 90.)



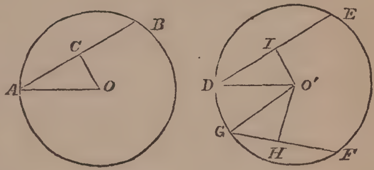
1. Let C and C' be the centres of two circles which intersect in A and B. Draw AC, AC'. Then (Ax. 10, Cor.), $CC' < AC + AC'$, or than the sum of the radii. Also, $CC' + AC' > AC$, and transposing, $CC' > AC - AC'$.

2. In the figure belonging to the last theorem draw the line AB, connecting the two points of intersection. The two points A and B, being both on the circumference whose centre is C, are equally distant from C, and also, being on the circumference whose centre is C', they are equally distant from C'; hence (Bk. I., Th. XIV., Cor. 3), AB is perpendicular to CC', and AB, being a chord of either circle, is bisected by the radius perpendicular to it.

3. In the two circles ABD and EFG let the arc EF be greater than the arc AB. Draw the radii AC, BC, EH, and FH. These radii, being drawn in equal circles, will be equal, but the angle ACB will be less than EHF, because it is subtended by a smaller arc. Then (Bk. I., Th. VIII.), since the triangles EHF and ABC have two sides of each equal and the included angles unequal, the side EF will be greater than the side AB.



Again, if the chord AB is greater than chord EF, the triangle ABC and EHF have two sides equal, each to each, and the third sides unequal; hence (Bk. I., Th. VIII., Cor.), the angle EHF is greater than the angle ACB, and therefore its measure, the arc EF, is greater than AB, the measure of ACB.



4. In the two circles ABM and DEF, let the chord AB equal the chord DE. In the circle ABM draw radius OA and perpendicular OC, and in DEF draw $O'D$ and $O'I$. Then, in the two right-angled triangles ACO and DIO', $OA = O'D$, $AC = DI$, being halves of equal chords; hence, $OC = O'I$.

Again, let GF be a chord less than AB or DE, and draw the perpendicular $O'H$ and radius $O'G$. The two right-angled triangles DIO' and GHO' have the hypothenuses $O'G$ and $O'D$ equal, but the side GH, which is half the chord GF, is less than DI, half of chord DE. But (Bk. III., Th. VI., Cor. 1), $\overline{O'H}^2 = \overline{O'G}^2 - \overline{GH}^2$ and $\overline{O'I}^2 = \overline{O'D}^2 - \overline{DI}^2$; hence, $O'I$ will be less than $O'H$, if DE is greater than GF.

5. The diagonal of an inscribed square is the diameter of the circle. From Bk. III., Th. VI., Cor. 3, we have, referring to the figure,

$$AB : AC :: 1 : \sqrt{2};$$

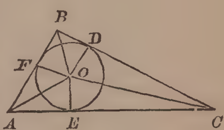
dividing correspondents by 2, and inverting

$$\frac{AC}{2}, \text{ or } R : AB :: \tfrac{1}{2}\sqrt{2} : 1;$$

multiplying second ratio by $\sqrt{2}$,

$$R : AB :: 1 : \sqrt{2}.$$

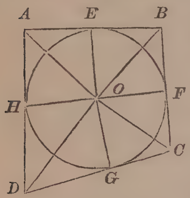
6. See Problem XVII., p. 99.



7. Let ABC be any triangle, and inscribe in it a circle. From the centre O draw lines to the vertices of the triangle, and also radii to the points of contact with the sides. The radii OD, OE, and OF are perpendicular to the sides of the triangle (Th. IV., Cor.). The area of the triangle OAC is equal to $AC \times \frac{1}{2}OE$, of OAB to $AB + \frac{1}{2}OF$ and of BOC to $BC \times \frac{1}{2}OD$. But these three triangles make up the triangle ABC; hence, area of $ABC = (AC + BC + AB) \times \frac{1}{2}OE$.

8. Taking the figure Th. VII., Cor. 2, we have the quadrilateral

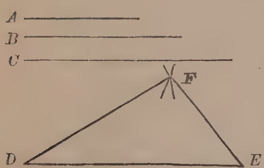
$ADCE$, having the angle ADC measured by half the arc AEC, and the angle AEC measured by half the arc ADC. But these two arcs are equal to the whole circumference; hence, the angles ADC and AEC are measured by half a circumference, and are consequently equal to two right angles; and the same may be proved of the other angles.



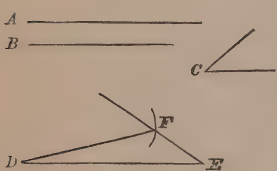
9. Let $ABCD$ be a quadrilateral circumscribed about a circle. From the centre O draw the radii OE, OF, OG, and OH to the points of contact, and the lines OA, OB, OC, and OD to the vertices of the quadrilateral. The two right-angled triangles AOE and AOH have $OE=OH$ and OA common; hence, $AE=AH$. In the same way we may prove $EB=BF$, $CG=CF$, $DG=DH$. Uniting these equations, we have $AE+EB+CG+DG=AH+DH+BF+CF$, or $AB+CD=AD+BC$.

10. The expression for the semi-circumference is πR; the expression for the area πR^2. Making $R=1$, both expressions become π.

PROBLEMS FOR ORIGINAL THOUGHT. (p. 99.)



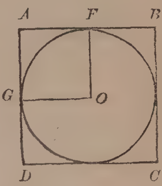
1. Let A, B, and C be the given sides. Draw DE, making it equal to C. Then, with D as a centre and a radius equal to B, describe an arc; from E as a centre, with a radius equal to A, describe another arc; from their point of intersection, F, draw DF and EF; then will DEF be the required triangle.



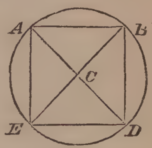
2. Let A and B be the given sides, and C the given angle opposite B. Draw DE equal to A, and at either end, as E, lay off an angle equal to C, producing the side indefinitely. From D as a centre, with a radius equal to B, describe an arc cutting the indefinite line in F; join DF, and DEF will be the required triangle.

3. Let ABC (Th. for Orig. Thought, 7) be the given triangle. Bisect the angles A and B by the lines AO and BO, and from their point of intersection, O, let fall perpendiculars upon the sides of the triangle; a circle drawn through the three points of contact will be the inscribed circle. For, the two triangles AOE and AOF have the side AO common, the angle OEA equal to OFA, because both are right angles, and

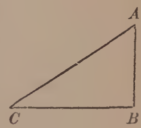
the angle OAF equal to OAE by construction; therefore OE is equal to OF. In the same way OF and OD may be shown to be equal. Hence, a circle with radius OE will touch the triangle at the points D, E, and F, and be inscribed in it.



4. Let $ABCD$ be any square, and from the middle points of two sides, as AB and AD, let fall perpendiculars; then their point of intersection, O, will be the centre of the circle, and the perpendiculars the radii. For, the quadrilateral $AFGO$ has three of its angles right angles by construction; hence, GOF is a right angle, and since the adjacent sides AF and AG are equal, their opposite sides OF and OG must be equal. These lines by construction are also perpendicular to the tangent of the circle; hence, they must be radii of the circle.

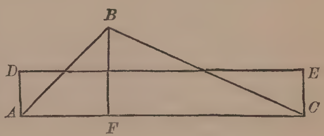


Again, in the circle $ABDE$ draw any diameter, as AD, and draw BE perpendicular to it. Join the extremities of these diameters by lines, and the figure thus formed will be the inscribed square. For, the two right-angled triangles AEC and ACB have two sides equal; hence, AE and AB are also equal, and each angle of the quadrilateral, being inscribed in a semicircle, is a right angle; hence, the inscribed figure is a square.



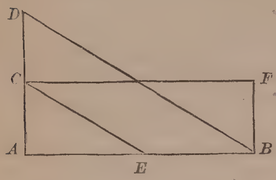
5. Let BC be the side of one of the given squares. At B erect a perpendicular AB equal to the side of the other given square. Join A and C, and AC will be the side of the required square. For, $\overline{AC}^2 = \overline{BC}^2 + \overline{AB}^2$. (Bk. III., Th. VI.)

6. In the same figure, let BC be the side of the lesser of the given squares, and at the point B erect a perpendicular. Then, with the point C as a centre and a radius equal to the side of the greater square, describe an arc till it cuts the perpendicular in some point, as A. Then, AB will be the side of the required square. For (Bk. III., Th. VI., Cor. 1), $\overline{AC}^2 - \overline{BC}^2 = \overline{AB}^2$.



7. Let ABC be the given triangle and BF its altitude. The area of ABC equals $\frac{1}{2}BF \times AC$. Through the middle point of BF pass a line parallel to AC, and at A and C erect perpendiculars, cutting the parallel line in D and E. Then, the area of the rectangle $ACED$ equals $\frac{1}{2}BF \times AC$; hence, it is equal to the area of the triangle ABC.

8. In the figure to Problem XIV. take two indefinite lines meeting at any angle, and lay off on one AD equal to the first of the given lines and DF equal to the second. On the other indefinite line lay off AC equal to the third line, and draw CD. Then from the point F draw EF parallel to CD, and EC will be the fourth proportional required. For (Bk. III., Th. IX.), $AD : DF :: AC : CE$.



9. Let AB be the line upon which the required rectangle is to be constructed. At A erect a perpendicular to AB, and lay off AD equal to the longer side of the given rectangle, and AE equal to the shorter side. Join B and D, and through E draw a line parallel to BD. Then we have the proportion (Bk. III., Th. IX., Cor. 1),

$$AB : AE :: AD : AC;$$

or (Bk. II., Th. I.), $\quad AB \times AC = AE \times AD.$

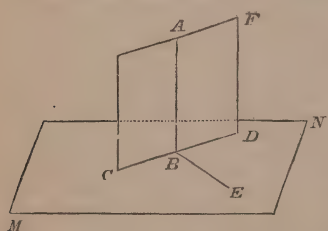
But $AD \times AE =$ area of given rectangle; hence, completing the rectangle $ACFB$, we have the required rectangle.

10. Find a mean proportional (Problem XV.) between the base and altitude of the parallelogram, and on it, as a side, construct a square; it will be the square required.

BOOK V.

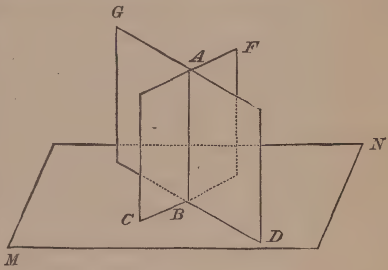
THEOREMS FOR ORIGINAL THOUGHT. (p. 108.)

1. From the given point let fall a perpendicular upon the given line; the plane must pass through this perpendicular (Def. 2). At their point of intersection erect another perpendicular to the given line; the plane must pass through this perpendicular (Th. III.), and only one plane can be passed through these two lines.



2. Let AB be perpendicular to the plane MN, and pass the plane CF through AB. Then in the plane MN draw BE perpendicular to CD, the intersection of CF and MN. Since AB is perpendicular to MN, it is perpendicular to CD and BE (Def. 2), and ABE measures the diedral angle of the planes (Def. 6); and since ABE is a right angle, the planes are perpendicular to each other.

3. Let the two planes in the last figure be perpendicular to each other, and the line AB, in the plane CF, be perpendicular to their intersection CD; and draw the line BE perpendicular to CD at B. Then, since the planes are perpendicular to each other, the angle ABE is a right angle; hence, AB is perpendicular to both CD and BE at their intersection, and therefore perpendicular to their plane MN.



4. Let GD and CF be two planes perpendicular to the plane MN, one of the points of intersection being B. At B erect a perpendicular to MN. Since the plane CF is perpendicular to MN, a line may be drawn in that plane perpendicular to MN at the point B. But AB must be that perpendicular, or there would be two perpendiculars to the plane at the same point, which is impossible. In the same way it may be shown that AB must be in the plane GD; hence, it is the intersection of the two planes.

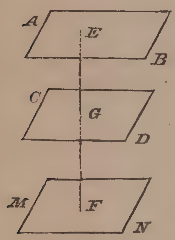
5. Taking figure of Theorem 2, let CD be the given line in the given plane MN, and suppose two planes passed through it perpendicular to MN. Draw BE in the plane MN perpendicular to CD. If the planes are perpendicular to MN, the diedral angles must be right angles, or the perpendiculars in those planes to CD, erected at B, must also be perpendicular to BE, and hence perpendicular to the plane MN (Th. III.). But this would admit of two perpendiculars to a plane at the same point, which is impossible; hence, but one plane can be passed.

6. Suppose two planes to be passed through the given line perpendicular tc the given plane. From any point of the given line draw lines in the supposed planes perpendicular to their intersection with the given plane; then will these lines be perpendicular to the given plane (Th. 3). But they must then be parallel to each other (Th. IV.), which is impossible, since they meet. Hence, there can be but one plane.

7. If the intersection is not parallel to the given lines, from any point of the intersection draw in the plane of one of the lines a line parallel to that line. From the same point draw in the other plane a line parallel to the other given line. Now, since each of these lines is parallel to one of the given parallel lines, they must be parallel to each other, since they have the same direction. But they meet in a given point; there-

fore to have the same direction they must coincide with each other and with the intersection, since that is the only line in both planes.

8. In the plane to which the given line is parallel, draw a line parallel to the given line. Then (Th. IV.), the line just drawn is perpendicular to the other plane. But (Th. 2, Original Thought) the plane in which the line is drawn must be perpendicular to the other plane.



9. Let AB and CD be two planes parallel to MN. At F erect a perpendicular to MN, then (Th. VII.) it will be perpendicular to AB and CD; hence (Th. V.), AB and CD are parallel to each other.

10. If possible, pass two planes through the given point parallel to the given plane, and draw a line from the given point perpendicular to the given plane; this line will also be perpendicular to the other planes (Th. VII.); and drawing lines in these planes at right angles to the perpendicular at the given point, we shall have two perpendiculars in the same plane (for these two lines determine a plane) drawn to a line at the same point, which is impossible. Hence, it is impossible to pass two planes through the point parallel to the given plane. Or, it may be demonstrated from the previous theorem thus: If two planes are drawn through the given point parallel to the given plane, they will be parallel to each other, but two parallel planes cannot pass through the same point; hence, there can be but one plane.

11. If the given lines are perpendicular to the third plane, the planes may intersect, as may be seen by referring to Theorems 4 and 7. But if the lines are not perpendicular to the third plane, and this is the intention of the theorem, then, according to Theorem 4, if their planes intersected, the intersection must be perpendicular to the third plane, and by Theorem 7, at the same time parallel to the given lines; but as these two conditions are inconsistent, the planes cannot intersect, and are therefore parallel. Hence, the proposition is true when the given lines are not perpendicular to the third plane.

PROBLEMS. (p. 109.)

1. Draw any line AB in a given plane MN, and pass any plane through it. Then from the given point in the given plane draw a perpendicular to the line AB, and at the point of intersection erect a perpendicular in the second plane to the line AB. A plane passed through these two perpendiculars will be perpendicular to the given plane. There-

fore, if at the given point we erect a perpendicular in the third plane, it will be the required perpendicular (Th. 3).

2. Having erected a perpendicular to the given plane at any point, we take some point of this line, and there erect two perpendiculars. Through the last two lines pass a plane; it will be parallel to the given plane (Th. V.).

3. At any point of the given intersection erect a perpendicular to the plane. Then pass a plane through these two lines, and it will be the required plane. For, the plane will be perpendicular to the given plane (Th. 2), and there can be but one plane passed through the lines (Th. I., Cor. 2).

4. Draw any line in the given plane passing through the given point. Through this line pass a plane perpendicular to the given plane. Construct an angle in this plane equal to the given angle, taking the intersection as a base (Prob. IV.) and the given point as a vertex. The angle thus constructed is the required angle.

5. Draw any line in the given plane, and through it pass a plane perpendicular to the given plane. In the second plane, taking the intersection as a base, construct the given angle. Through the vertex of the angle, in the given plane, draw a line perpendicular to the sides of the angle. If we pass a plane through this last line, and the side of the angle lying in the second plane, the angle will be the diedral angle of the plane, and hence the plane will be inclined at the required angle.

NOTE.—These constructions cannot actually be made without a knowledge of Descriptive Geometry and Perspective, but the principles involved in them can readily be seen.

BOOK VI.

PRACTICAL EXAMPLES. (p. 128.)

1. The convex surface $= 16 \times 14 = 224$ square inches.

2. The contents $= 24 \times 7 = 168$ cubic feet.

3. The convex surface $= 6 \times 5 = 30$; $30 \times \frac{18}{2} = 270$ square inches.

4. Volume of first pyramid $= 8^2 \times \frac{12}{3} = 256$; volume of 2d pyramid $= 6^2 \times \frac{12}{3} = 144$; base of 3d pyramid $= \sqrt{8^2 \times 6^2} = 48$; volume of 3d pyramid $= 48 \times \frac{12}{3} = 192$; $256 + 144 + 192 = 592$ cubic inches, volume of frustum.

5. Each side $=11^2=121$; six sides $=121\times 6=726$ square inches.

6. One edge $=\sqrt[3]{373248}=72$; one side $=72^2=5184$ square inches; entire surface $=5184\times 5=25920$ square inches $=180$ square feet.

7. $20.5\times 10\frac{2}{3}\times 6.75=1476$ cubic feet; $\sqrt[3]{1476}=11.38+$.

8. $1600\times 231=369600$ cubic inches $=213.8$ cubic feet; $\sqrt[3]{213.8}=5.98+$.

9. Let AB represent a side of the cube; then the contents $=\overline{AB}^3$, and the surface $=6\times\overline{AB}^2$; hence, $\overline{AB}^3=6\ \overline{AB}^2$, for which we have $AB=6$.

10. Let P and P' represent the prisms, and we have $P:P'::7^3:28^3$, or $1^3:4^3$ or $1:64$.

11. A regular hexagon may be divided into six equal equilateral triangles, each side being 6 inches. Letting fall a perpendicular from the vertex of any angle in one of these triangles, it will bisect the base, and we shall have (Bk. III., Th. VI., Cor. 1) altitude of triangle $=\sqrt{6^2-3^2}$ $\sqrt{27}=5.196152$; area of one triangle $=\dfrac{6\times 5.196152}{2}=15.588456$; area of hexagon $=15.588456\times 6=93.530736$; volume of pyramid $=\dfrac{93.530736\times 20}{3}$ $=623.5382+$ cubic inches.

12. Since, by Th. IX., the edges are divided proportionally, the triangle forming one side of the pyramid has its sides divided proportionally by a line parallel to the base (Bk. III., Th. IX., Cor. 2), and hence the side of the upper base of the frustum is equal to half the side of the lower base. To find the area of the upper base, we have altitude of triangle $=\sqrt{3^2-1.5^2}=\sqrt{6.75}=2.598076$; area of triangle $=\dfrac{3\times 2.598076}{2}$ $=3.897114$; area of upper base $=3.897114\times 6=23.382684$. Area of mean proportional $=\sqrt{93.530736\times 23.382684}=\sqrt{4\times 23.382684^2}=46.765368$. Then, volume of frustum $=(93.530736+23.382684+46.765368)\times\frac{10}{3}=$ $(7\times 23.382684)\times\frac{10}{3}=545.59596$ inches.

The slant height of the pyramid is the hypothenuse of a triangle of which the perpendicular is the altitude of the pyramid, and the base the altitude of one of the triangles composing the base. Hence, slant height of pyramid $=\sqrt{20^2+5.196^2}=20.663$ inches. The plane cutting the pyramid makes the slant height of frustum $=\dfrac{20.663}{2}=10.331+$ inches. Then, convex surface of frustum $=\dfrac{(6\times 6+3\times 6)}{2}\times 10.331=278.937$ square inches.

13. The contents $=2150.42\times 100=215042$ cubic inches $=124.4456$ cubic feet; $\sqrt[3]{124.445}=4.9+$ feet.

THEOREMS FOR ORIGINAL THOUGHT. (p. 129.)

1. Since (Th. VI.) the volume of any parallelopipedon is equal to the product of its base and altitude, when the bases and altitudes are respectively equal their product must be equal, and hence the volumes are equal.

2. It is shown (Th. IV., Cor. 3) that the square of either diagonal equals the sum of the squares of the three edges which meet at the same vertex. But these three edges in a rectangular parallelopipedon are the length, breadth, and thickness of the parallelopipedon; hence, the three edges will be the same whatever vertex we take, and the diagonals, which are the square root of the sum of the squares of the edges, must also be equal.

3. The plane passing through the opposite edges of the parallelopipedon cuts the base diagonally and divides it into equal triangles (Bk. I., Th. XV., Cor. 1); the altitude of the prisms is the same as that of the parallelopipedon; hence (Th. III.), the prisms are equal.

4. Each prism is equal to the product of its base and altitude; and since the bases are equal, they will be to each other as their altitudes.

5. In each pyramid let fall a perpendicular from the vertex upon the centre of the base, and join the point of intersection with one of the vertices of the base. In the right-angled triangles thus formed we have the hypothenuses and angles respectively equal by construction; hence, the sides must also be equal and the two pyramids have equal altitudes. But pyramids having equal bases and equal altitudes are equal (Th. XI.).

6. Since, in the faces of the polyedron the homologous sides are respectively proportional, and every edge belongs to two faces, the ratio between any two homologous edges can be shown to be the same as between any other two edges. But any two homologous faces are to each other as the squares of their homologous edges; and since the ratio between the homologous edges is the same, the ratio of their squares must be the same. Hence, representing the edges of the polyedrons by A, B, C, D, etc., and a, b, c, d, etc., we have $A^2 : a^2 :: B^2 : b^2 :: C^2 : c^2 :: D^2 : d^2$, etc. Letting S and s, and S' and s', S'' and s'' represent the faces of the polyedrons, we have from the preceding proportion,

$$S : s :: A^2 : a^2.$$
$$S' : s' :: A^2 : a^2,$$
$$S'' : s'' :: A^2 : a^2.$$

Hence, combining the proportions (Bk. II., Th. VIII.),

$$S : s :: S' : s' :: S'' : s'', \text{ etc.};$$

and (Bk. II., Th. XII.),

$$S+S'+S'', \text{ etc.} : s+s'+s'', \text{ etc.} :: S : s.$$

But, $S : s :: A^2 : a^2$; hence,

$$S+S'+S'', \text{ etc.} : s+s'+s'', \text{ etc.} :: A^2 : a^2.$$

$S+S'+S''$, etc., compose the convex surface of one polyedron and $s+s'+s''$, etc., of the other; hence, the convex surfaces are to each other as the squares of their homologous edges.

BOOK VII.

PRACTICAL EXAMPLES. (p. 144.)

1. Circumference of base $= 3.1416 \times 8 = 25.1328$; convex surface of cylinder $= 25.1328 \times 16 = 402.1248$ square inches. Area of base $= 3.1416 \times 4^2 = 50.2656$; contents of cylinder $= 50.2656 \times 16 = 804.2496$ cubic inches.

2. Circumference of base $= 3.1416 \times 20 = 62.832$; slant height $= \sqrt{24^2+10^2} = 26$; convex surface $= 62.832 \times \frac{26}{2} = 816.816$ square inches. Area of base $= 3.1416 \times 10^2 = 314.16$; contents $= 314.16 \times \frac{24}{3} = 2513.28$ cubic inches.

3. Circumference of upper base $= 3.1416 \times 12 = 37.6992$; circumference of lower base $= 3.1416 \times 42 = 131.9472$. The slant height is the hypothenuse of a triangle whose perpendicular is the altitude of the frustum, and whose base is the difference between the radii of the upper and lower bases; hence, slant height $= \sqrt{36^2+15^2} = 39$; convex surface $= \frac{1}{2}(37.6992 + 131.9472) \times 39 = 3308.1048$ square inches.

4. Area of lower base $= 3.1416 \times 4 = 12.5664$ sq. feet; area of upper base $= 3.1416 \times 1^2 = 3.1416$ sq. ft.; mean proportional $= \sqrt{3.1416 \times 4 \times 3.1416 \times 1} = 6.2832$ square feet; volume $= (12.5664 + 3.1416 + 6.2832) \times \frac{9}{3} = 65.9736$ cubic feet.

5. Surface $= 4\pi R^2 = 3.1416 \times 16^2 = 804.2496$ square inches; contents $= \frac{1}{6}\pi D^3 = \frac{1}{6} \times 3.1416 \times 16^3 = 2144.6656$ cubic inches.

6. Surface $= 4\pi R^2$; $\therefore R^2 = \dfrac{\text{surface}}{4\pi} = 1809.5616 \div 12.5664 = 144$; $R = 12$ inches and $D = 24$ inches. Volume $= 4\pi R^2 \times \frac{1}{3}R = 1809.5616 \times 4 = 7238.2464$ cubic inches.

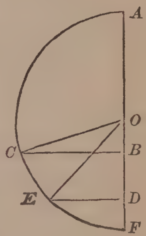
7. Volume $= \frac{1}{6}\pi D^3$; $\therefore D^3 = \dfrac{\text{volume}}{\frac{1}{6}\pi} = 113.0976 \div .5236 = 216.$ $D = \sqrt[3]{216} = 6$ inches; surface $= \pi D^2 = 3.1416 \times 36 = 113.0976$ sq. inches.

8. The altitude of the cylinder equals the diameter of the sphere; but $D = \sqrt[3]{\text{volume} \div \frac{1}{6}\pi} = \sqrt[3]{268.0832 \div .5236} = 8$ inches.

9. Surface of zone equals circumference of great circle multiplied by its altitude; hence, surface zone $= 3.1416 \times 100 \times 10 = 3141.6$ square feet.

10. Surface of zone forming base of sector $= 3.1416 \times 8 \times 2$. Volume of sector $= 3.1416 \times 8 \times 2 \times \frac{4}{3} = 67.0208$ cubic feet. Radius of base of cone is base of a right-angled triangle, radius of sphere being hypothenuse and difference between radius and altitude of segment being perpendicular of triangle; hence, radius of base $= \sqrt{4^2 - (4-2)^2} = \sqrt{12}$. Volume of cone $= 3.1416 \times 12 \times \frac{2}{3} = 25.1328$ cubic feet; $67.0208 - 25.1328 = 41.888$ cubic feet.

11. In the figure, let ACF be a semicircle and $CEBD$ the given segment, O being centre of sphere, $\overline{OC}^2 = \overline{BC}^2 + \overline{OB}^2$ and $\overline{OE}^2 = \overline{DE}^2 + \overline{OD}^2$. Since OC and OE are radii, we have $\overline{BC}^2 + \overline{OB}^2 = \overline{DE}^2 + \overline{OD}^2$. Substituting their values, we have $BC = \frac{24}{2}$, $DE = \frac{20}{2}$, $BD = 4$; and hence, letting OB be represented by x, we have $OD = 4 + x$, and the equation becomes $144 + x^2 = 100 + 16 + 8x + x^2$. Collecting, $8x = 28$ and $x = 3\frac{1}{2}$; hence, $\overline{OC}^2 = 144 + 12.25$ and radius $= 12\frac{1}{2}$ inches. Then, zone $CE = 3.1416 \times 25 \times 4 = 314.16$ and segment $CEBD =$ zone $CE \times \frac{1}{3}OC + \pi\overline{ED}^2 \times \frac{1}{3}OD - \pi\overline{BC}^2 \times \frac{1}{3}OB = 314.16 \times \dfrac{12\frac{1}{2}}{3} + 3.1416 \times 100 \times \dfrac{7\frac{1}{2}}{3} - 3.1416 \times 144 \times \dfrac{3\frac{1}{2}}{3} = 1566.6112$ cubic inches.



THEOREMS FOR ORIGINAL THOUGHT. (p. 144.)

1. The line of intersection of two great circles is evidently a diameter of each, since their centre is common, and therefore the line of intersection must pass through it. But a diameter divides the circle into two equal parts; hence, the circles bisect each other.

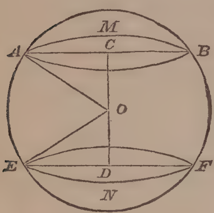
2. The parts of the sphere may be so placed as to coincide; for if they do not, there would be some points unequally distant from the centre, which is impossible.

3. If we let fall a perpendicular from the centre of the sphere upon the plane of the small circle, as in figure to Th. VII., and draw the lines

OD and OE from the point of intersection to different points in the circumference of the small circle, and draw CD and CE, radii of the sphere, then the two right-angled triangles COD and COE have two sides equal; hence, the sides OD and OE must be equal (Bk. III., Th. VI., Cor. 4), and therefore are radii of the small circle, and their point of intersection, O, is the centre of the small circle.

4. In figure to Th. VII., in the triangle COD, CD is a radius of the sphere and OD a radius of the small circle. But the triangle is right-angled at O; hence, CD, being the hypothenuse, must be greater than OD.

5. Let two small circles AMB and ENF be passed through a sphere equidistant from the centre O. Draw OC and OD perpendicular to the diameters of the circles, and the radii AO and EO to the extremities of the diameters. The right-angled triangles ACO and EDO have the side OD = side OC by hypothesis, and $EO = AO$, being both radii of the sphere; hence, the remaining sides AC and ED are also equal. But (Th. 3) C and D are the centres of the small circles; hence, the lines AC and ED are their radii, and the circles are equal.



6. Taking the figure in Th. 1 under Bk. IV., through C and C', the centres of the spheres, let any plane be passed, cutting the spheres in great circles intersecting in A and B; then (Th. 2, Bk. IV.), the line CC' joining the centres will bisect AB, joining the points of intersection, at right angles. If we revolve the circles round the line CC', they will generate the spheres, and the line AB, revolved around its middle point, will generate a circle.

7. The two given points, together with the centre of the sphere, determine the position of a plane (Bk. V., Th. I., Cor. 1), and the section made by a plane passing through the centre must be a great circle.

8. A cone inscribed in a cylinder has the same base and altitude; hence (Th. IV., Cor. 2), the cone is to the cylinder as 1 to 3. A sphere inscribed in a cylinder is to the cylinder as 2 to 3 (Th. XI.); hence, the three bodies are as 1, 2, 3.

Or, letting R represent the radius of the sphere, we have cone $= \frac{2}{3}\pi R^3$, sphere $= \frac{4}{3}\pi R^3$, cylinder $= 2\pi R^3$; hence, cone : sphere : cylinder : : $\frac{2}{3} : \frac{4}{3} : 2$, or $1 : 2 : 3$.

MISCELLANEOUS PROBLEMS.—PLANE FIGURES. (p. 145.)

1. Surface of 1 brick $=32$ inches; surface of yard $=16\times20\times144=46080$ cubic inches; $46080\div32=1440$.

2. Surface of wall $=(24+18)\times2\times12=1008$; surface of ceiling $=24\times18=432$; number of square feet $=1008+432=1440$; $1440\div9=160$ square yards; $160\times\$0.16=\$25.60$.

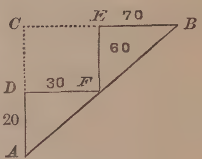
3. Area of rectangle $=60\times40=2400$ square feet. Side of square $=\dfrac{(60+40)\times2}{4}=50$; area of square $=50^2=2500$; $2500-2400=100$ square feet.

4. Area of square $=16\times25=400$; side of square $=\sqrt{400}=20$; diagonal of square $=20\times\sqrt{2}=28.28$ inches.

5. Diagonal $=$ side $\times\sqrt{2}$; $\therefore$ side $=\dfrac{\sqrt{50}}{\sqrt{2}}=5$ inches.

6. Diagonal of rectangular parallelopipedon $=\sqrt{48^2+20^2+39^2}=65$ feet.

7. Drawing a diagram and taking A for the starting point and B for the point reached, we see that producing AD till the part produced equals EF, and BE till the part produced equals DF, and joining A and B, we have a right-angled triangle. Then, $(70+30)^2+(20+60)^2=\overline{AB}^2$; hence, $AB=128.06$ miles.



8. We have a right-angled triangle of which half the width of the gable is the base, and the height of the ridge-pole the perpendicular; hence, $\sqrt{24^2+10^2}=26$ feet, the length of rafter.

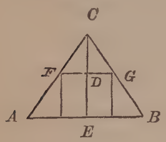
9. Altitude $=\sqrt{15^2-10^2}=11.1803$; area $=\dfrac{20\times11.1803}{2}=111.803$.

10. The broken part forms the hypothenuse of a right-angled triangle; hence, hypothenuse $=\sqrt{48^2+36^2}=60$; whole length $=36+60=96$ feet.

11. Let $x=$ length of a side; then, the altitude $=\sqrt{x^2-\dfrac{x^2}{4}}=\dfrac{x}{2}\sqrt{3}$.

Area $=\dfrac{x^2}{4}\sqrt{3}=272\frac{1}{4}$; $x^2=\dfrac{1089}{\sqrt{4}}$; $x=25.076$ square feet.

12. Area $=\pi R^2$; $\therefore R=\sqrt{\frac{19.635}{3.1416}}=2.5$; $D=5$ inches; circumference $=5\times 3.1416=15.708$ inches.



13. Altitude of triangle $=\sqrt{16^2-8^2}=13.856$. Then, by similar triangles, $AB:FG::EC:DC$, or representing side of square by x, and substituting values, $16:x::13.856:13.856-x$. Multiplying means and extremes together, $13.856x=16\times 13.856-16x$, whence, $x=7.425$.

14. The area of a trapezoid equals one-half the sum of its parallel sides multiplied by the altitude; hence, sum of parallel sides $=\frac{15}{12}\times 2=\frac{5}{2}$ feet $=30$ inches; $2+3=5$; hence, one is $\frac{2}{5}$ of 30, or 12 inches wide, and the other $\frac{3}{5}$ of 30, or 18 inches.

15. Since 40 minutes is $\frac{2}{3}$ of an hour, the minute-hand would pass over $\frac{2}{3}$ of the area of the circle, which is $\frac{2}{3}(3.1416\times 6^2)=75.398$ square inches.

16. Side of square $=\sqrt{25}=5$; diagonal of square $=5\sqrt{2}=7.07$; circumference $=3.1416\times 7.07=22.211112$.

17. If they go 30 miles an hour, they go half a mile a minute; half a mile $=2640$ feet; $2640\div 200=13\frac{1}{5}=$ circumference of wheel; diameter $=13\frac{1}{5}\div 3.1416=4\frac{1}{5}+$ feet.

18. The rope is the hypothenuse and the post the perpendicular of a right-angled triangle, the base being the radius of the circle. Then, $400-36=364=$ square of radius; area $=\frac{3.1416\times 364}{9}=127.06$.

19. In this circle $\pi R^2=2\pi R$, or $R^2=2R$; hence, $R=2$; area $=3.1416\times 4=12.5664$.

20. $R=\sqrt{\frac{4\times 160}{3.1416}}=14.27$; circumference $=28.54\times 3.1416=89.66$ rods; 6 miles an hour $=32$ rods a minute; $89.66\div 32=2$ minutes 48 seconds.

21. Radius of whole garden $=\sqrt{\frac{2\times 160}{3.1416}}=10.092$; $\frac{3}{4}$ of 2 acres $=240$ rods; radius of inner circle $=\sqrt{\frac{240}{3.1416}}=8.740$; $10.092-8.740=1.352$ rods $=22.308$ feet.

22. The circle described by the hour-hand has an area of 16×3.1416 square inches; but in an hour the hand passes over $\frac{1}{12}$ of this area, or $\frac{4}{3}\times 3.1416$ square inches. The area of the circle described by the minute-hand in the same time is 36×3.1416 square inches. The difference of these areas $=34\frac{2}{3}\times 3.1416=108.9088$ square inches.

MISCELLANEOUS PROBLEMS.—VOLUMES. (p. 146.)

1. Top and bottom $=8\times4\times2=64$; sides $=8\times2\times2=32$; ends $=4\times2\times2=16$; $64+32+16=112$ square inches.

2. Convex surface $=4\times4\times\frac{12}{2}=96$; base $=4\times4=16$; entire surface $=96+16=112$ square inches.

3. Convex surface $=3.1416\times12\times16=603.1872$; area of the bases $=3.1416\times36\times2=226.1952$; entire surface $=603.1872+226.1952=829.3824$ square inches.

4. Slant height $=\sqrt{16^2+12^2}=20$; convex surface $=3.1416\times24\times10=753.984$; area of base $=3.1416\times144=452.3904$; entire surface $=753.984+452.3904=1206.3744$ square feet.

5. Edge of the cube equals diameter of sphere. Surface of sphere $=4\pi R^2=3.1416\times400=1256.64$; volume of sphere $=\frac{1}{6}\pi D^3=\frac{1}{6}(3.1416\times8000)=4188.8$; contents of cube $=20^3=8000$; space between $=8000-4188.8=3811.2$ cubic inches.

6. Surface $=\pi D^2$; $\therefore D=\sqrt{\dfrac{6.305}{3.1416}}=1.416664$; volume $=\frac{1}{6}(6.305\times1.416664)=1.488677$.

7. Volume $=\frac{1}{6}\pi D^3$; $D=\sqrt[3]{\dfrac{6\times1.2411}{3.1416}}=15.999$ inches; surface $=256\times3.1416=804.2496$ square inches $=5.58506$ square feet.

8. Circumference of base $=116.666\div14=8.333\frac{2}{7}$; diameter $=8.333\frac{2}{7}\div3.1416=2.65$.

9. Diameter $=20\div3.1416$; area $=\dfrac{(20\div3.1416)\times20}{4}=\dfrac{100}{3.1416}$; volume $=\dfrac{100\times20}{3.1416}=636.61$ feet. By using the decimal given in the last rule for the area of a circle (page 167), the exact answer given in the book may be obtained.

10. Area of base $=3.1416\times1=3.1416$; $15.708\div3.1416=5$ feet.

11. Circumference of base $=3.1416\times4.5=14.1372$; $141.372\div14.1372=10=$ half slant height; slant height $=20$ ft.; altitude $=\sqrt{20^2-(\frac{1}{2}\times4.5)^2}=\sqrt{394.9375}=19.87$ feet.

12. Since the circumferences of circles are as their radii (Bk. IV., Th. XI.), we have by similar triangles (Bk. III., Th. IX., Cor. 1), $30:6::10:x$; whence, $x=2$, circumference of upper base. Convex surface $=\frac{1}{2}(10+2)\times24=144$ square feet.

13. $376.992 \div 20 = 18.8496 = \frac{1}{2}$ sum of circumferences; $\frac{4 \times 3.1416}{2} = 6.2832 =$ half of lesser circumference; $18.8496 - 6.2832 = 12.5664 =$ half circumference of greater end; $12.5664 \times 2 \div 3.1416 = 8$, diameter of greater end.

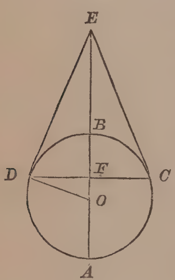
14. Area of base $= 8.83575 \div \frac{15}{3} = 1.76715$; radius $= \sqrt{\frac{1.76715}{3.1416}} = .75$ ft.; diameter $= 1.5$ feet, or 18 inches.

15. Volume $= (3.1416 \times 2^2 + 3.1416 \times 1^2 + 3.1416 \times 2 \times 1) \times \frac{1}{3}$ altitude $= (3.1416 \times 7) \times \frac{1}{3}$ altitude; hence, altitude $= (65.9736 \div 3.1416 \times 7) \times 3 = 9$ feet.

16. Slant height $= \sqrt{5^2 + 12^2} = 13$; convex surface $= \frac{1}{2}(18 + 8)\pi \times 13 = 169\pi$; areas of bases $= (9^2 + 4^2)\pi = 97\pi$; entire surface $= 169\pi + 97\pi = 266\pi$.

17. The slant height is the hypothenuse of a triangle whose perpendicular is the altitude and base the difference of half the sides of the upper and lower bases. Hence, $(4\frac{1}{2} - 2) = 2\frac{1}{2}$; slant height equals $\sqrt{12^2 + 2.5^2} = 12.257$. Convex surface $= (4 \times 9 + 4 \times 4) \times \frac{12.257}{2} = 318.68$; areas of bases $= 81 + 16 = 97$; entire surface $= 318.68 + 97 = 415.68$ square feet.

18. Since (Bk. VII., Th. IX., Cor. 5) a zone is to the surface of a sphere as the altitude of the zone to the diameter of the sphere, on the diameter AB we lay off BF equal to $\frac{1}{3}$ of the diameter and draw the chord DC perpendicular to AB; then, the zone DBC will be $\frac{1}{3}$ of the surface of the sphere, and D and C will be in the required horizon. Draw DO and DE perpendicular to it, meeting AB produced in E, and E will be the required point and EB the required distance.



The two right-angled triangles DOF and DEO are similar, for they have the angle DOE common; hence, the angles are equal, and we have the proportion $EO : DO :: DO : FO$, or $\frac{1}{3}BO$; hence, $EO = \frac{DO^2}{\frac{1}{3}BO} = 3$ times radius, and $EB = EO - BO = 2$ times the radius.

SPHERICAL GEOMETRY.

PRACTICAL EXAMPLES. (p. 161.)

1. Area $=(A+B+C-2)\times T$; $A+B+C-2=\frac{70}{90}\times 3-2=\frac{1}{3}$ of right angle; $T=\frac{1}{8}(4\pi R^2)=\frac{1}{2}\pi R^2$; hence, area $=\frac{1}{3}\times\frac{1}{2}\pi R^2=\frac{1}{6}\pi R^2$.

2. $150°=\frac{150}{90}=\frac{5}{3}$ of right angle; area $=(S-2n+4)\times T=(\frac{5}{3}\times 6-12+4)\times\frac{1}{2}\pi R^2=\pi R^2$.

3. The sides of a polar triangle are the supplements of the angles of the given triangle; hence, $180°-80°=100°$; $180°-90°=90°$; $180°-140°=40°$.

4. $180°-75°=105°$; $180°-110°=70°$; $180°-130°=50°$.

5. Area $=(\frac{108}{90}+\frac{90}{90}+\frac{90}{90}-2)\times\frac{1}{2}\pi R^2=\frac{6}{5}\times\frac{1}{2}\pi R^2=\frac{3}{5}\pi R^2$.

6. Area $=(\frac{60}{90}+\frac{90}{90}+\frac{120}{90}-2)\times\frac{1}{2}\pi R^2=\frac{1}{2}\pi R^2$; $R=4$; $\frac{1}{2}\pi R^2=8\pi$.

7. Lune $=T\times 2A=90°\times T=\frac{1}{2}\pi R^2$.

8. Lune $=\frac{1}{2}\pi R^2\times\frac{108}{90}=\frac{3}{5}\pi R^2=15\pi$.

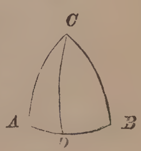
9. Volume $=L\times\frac{1}{3}R=\frac{144}{90}\times\frac{1}{2}\pi R^2\times\frac{1}{3}R=\frac{4}{15}\pi R^3$.

10. Volume $=\frac{72}{90}\times\frac{1}{2}\pi\times 5^2\times\frac{1}{3}$ of $5=16\frac{2}{3}\pi$.

11. Area $=\pi R^2=(A+B+C-2)\times T$; $\therefore$ $A+B+C-2=\dfrac{\pi R^2}{\frac{1}{2}\pi R^2}=2$; $A+B+C=4$ right angles; and as the angles are equal, each angle must equal $\frac{4}{3}$ of a right angle, or 120°.

12. Area of hexagon $=\left(\dfrac{130\times 6}{90}-12+4\right)\times\frac{1}{2}\pi R^2=\frac{1}{3}\pi R^2$; $(3A-2)\times T=\frac{1}{3}\pi R^2$; $3A-2=\dfrac{\frac{1}{3}\pi R^2}{\frac{1}{2}\pi R^2}=\frac{2}{3}$; $A=\frac{8}{9}$ of a right angle $=80°$.

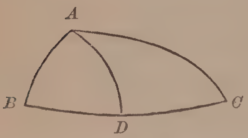
THEOREMS FOR ORIGINAL THOUGHT. (p. 162.)



1. Let ABC be an isosceles spherical triangle, having AC and BC equal; then, from C draw the arc of a great circle to the middle of the base AB. The two triangles ADC and BDC are mutually equilateral; hence (Th. IX. 3), they are mutually equiangular, and the angle A equals the angle B.

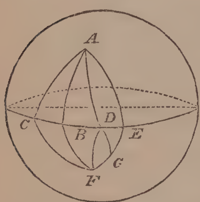
2. Let ABC, in figure of Th. VI., be the given triangle, angle C being equal to angle B, and let DEF be the polar triangle. Then, the sides DF and DE are equal (Th. VII.), and therefore the angles E and F are equal by the last theorem. But since the angles E and F are equal, the sides AC and AB must be equal in its polar triangle ABC.

3. Let ABC be a triangle having the angle A greater than angle B; then, BC will be greater than AC. Draw AD, making angle A equal to angle B; then, by the last theorem the triangle ABD is isosceles. But $AD+DC>AC$; or substituting for AD its equal BD, $BC>AC$.



Conversely, let BC be greater than AC, then A will be greater than B. For, if A were less than B, BC would be less than AC by the first part of the theorem, and if A and B were equal, the triangle would be isosceles; but both these conclusions are contrary to the hypothesis, and therefore absurd; therefore A must be greater than B.

4. It will be obvious that the two arcs will be continuous, as from any point, except a pole, only two perpendiculars can be drawn. Suppose, from the given point A, a perpendicular and oblique arc to be drawn meeting the circumference in C and B. Produce the perpendicular on the other hemisphere till $BF=AB$ and join CF. Then, since the angles ABC and CBF are right angles, the triangles ABC and CBF have two sides and an included angle equal, and CF therefore equals AC. But in the triangle ACF, $AC+CF>AF$ (Th. I.); hence, $AC>AB$.



5. Take a distance BD equal to BC and draw AD. Then, the triangles ABC and ABD have two sides and the included angle equal; hence, $AC=AD$.

6. Take a distance BE greater than BD and draw AE, and also the arcs DF and EF. Produce the arc AD till it cuts EF in some point, as G. Then, in the triangle AGE, $EG+AE>AG$, and in triangle DGF, $DG+GF>DF$. Adding GF to both members of the first inequality, and AD to both members of the second, we have $EF+AE>AG+GF$ and $AG+GF>AD+DF$; therefore, $EF+AE>AD+DF$. But it may be shown, as in Th. 4, that $AE=EF$ and $AD=DF$; whence, $AE>AD$.

7. The area of a spherical triangle $=(A+B+C-2)\times T$. Substituting the values, we have area $=(\frac{4}{3}\times 3-2)\times\frac{1}{2}\pi R^2$; and reducing, we have πR^2, which is the expression for the surface of a great circle.

MENSURATION.

MENSURATION OF LENGTHS AND SURFACES. (p. 164.)

THE TRIANGLE.

1. $\frac{40 \times 16}{2} = 320$ rods $= 2$ acres.

2. $20+30+40=90$; $\frac{90}{2}=45$; $45-20=25$; $45-30=15$; $45-40=5$; $45 \times 25 \times 15 \times 5 = 84375$; $\sqrt{84375} = 290.473+$ chains $= 29$ A. 7.56 P.

3. $150+200+250=600$; $\frac{600}{2}=300$; $300-150=150$; $300-200=100$; $300-250=50$; $300 \times 150 \times 100 \times 50 = 225000000$; $\sqrt{225000000} = 15000$; 15000 feet $= 1666.66$ square yards.

THE QUADRILATERAL.

PARALLELOGRAM.

1. $9 \times 7 = 63$ square feet.

2. $70.5 \times 70.5 = 4970.25$ square chains $= 497$ A. 4 P.

3. $333 \times 33.35 = 11105.55$ square feet $= 1233.95$ square yards.

TRAPEZOID.

1. 192 inches $= 16$ feet; 96 inches $= 8$ feet; $\frac{1}{2}(16+8) \times 12 = 144$ square feet.

2. $\frac{1}{2}(18+12) = 15$ inches $= 1\frac{1}{4}$ feet; $1\frac{1}{4} \times 24 = 30$ square feet.

3. $\frac{1}{2}(95+75) \times 65 = 5525$ rods $= 34$ A. 2 R. 5 P.

TRAPEZIUM.

1. $\frac{290 \times 60}{2} = 8700$; $\frac{290 \times 80}{2} = 11600$; $8700 + 11600 = 20300$ sq. inches $= 140$ square feet, 140 square inches.

2. $40+60+80=180$; $\frac{180}{2}=90$; $90-40=50$; $90-60=30$; $90-80=10$; $90 \times 50 \times 30 \times 10 = 1350000$; $\sqrt{1350000} = 1161.89$ = area of one triangle; $50+70+80=200$; $\frac{200}{2}=100$; $100-50=50$; $100-70=30$; $100-80=20$; $100 \times 50 \times 30 \times 20 = 3000000$; $\sqrt{3000000} = 1732.05$, area of second triangle; $1161.89 + 1732.05 = 2893.94$ square chains $= 289$ A. 1 R. 23 P.

POLYGONS.

REGULAR POLYGONS.

1. $\frac{1}{2}(14.6\times 6)\times 12.64=553.632$ square feet $=61.51466+$ square yards.

2. $\frac{9.941\times 8}{2}\times 12=477.168$ square feet.

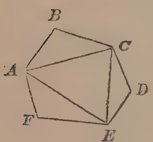
3. $2.5980762\times 5^2=64.9519050$ square inches.

4. $4.8284271\times(3\frac{1}{3})^2=53.64919$ square feet.

IRREGULAR POLYGONS.

1. Area of triangle $ABC=\frac{24\times 8}{2}=96$; area $ACD=\frac{18\times 10}{2}=90$; area of $AED=\frac{18\times 6}{2}=54$; $96+90+54=240$ square feet.

2. Let $ABCDEF$ be the given hexagon; then, drawing the diagonals as indicated in the problem, we find the hexagon to be made up of the four triangles ABC, CDE, AFE, ACE. In the triangle ABC, the half sum of sides $=\frac{268+249+459}{2}=488$; $(488-268)(488-249)(488-459)\times 488=744112160$; $\sqrt{744112160}=27278.41+$.



In the triangle CDE, we have $\frac{310+290+524}{2}=562$; $(562-310)(562-290)(562-524)\times 562=1463825664$; $\sqrt{1463825664}=38259.97+$.

In the triangle AFE we have $\frac{199+246+326}{2}=385.5$; $(385.5-199)(385.5-246)(385.5-326)\times 385.5=596752698.9375$; $\sqrt{596752698.9375}=24428.52+$.

In the triangle ACE we have $\frac{459+524+326}{2}=654.5$; $(654.5-459)(654.5-524)(654.5-326)\times 654.5=5485324166.4375$; $\sqrt{5485324166.4375}=74062.97+$. Adding these four areas, $27278.41+38259.97+24428.52+74062.97=164029.87$ square links $=1$ A. 2 R. 22 P. 13 yds. 49 ft.

THE CIRCLE.

THE CIRCUMFERENCE.

1. $3.1416\times 50=157.08$ inches.

2. $3.1416\times 32=100.5312$ rods.

3. $78.54\div 3.1416=25$ feet.

4. $3.1416\times 45=141.372$ rods.

LENGTH OF AN ARC.

1. *When Degrees and Radius are Given.*

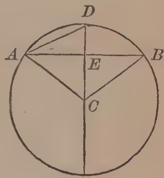
1. $.01745 \times 45 \times 10 = 7.8525$.

2. $32° \ 38' \ 42'' = 32.645°$; $32.645 \times .01745 \times 25 = 14.24138+$.

2. *When the Chord and Chord of half the Arc are Given.*

1. $(8 \times 60 - 96) \div 3 = 128$ inches.

2. Let AB be the given arc, and AC and CD radii of the circle, CD being perpendicular to the chord and AC meeting its extremity. Then, in the triangle AEC, $EC = \sqrt{\overline{AC}^2 - \overline{AE}^2} = \sqrt{100 - 64} = 6$; hence, $DE = 4$ and $AD = \sqrt{\overline{AE}^2 + \overline{ED}^2} = \sqrt{64 + 16} = 8.9442$. Then, $(8 \times 8.9442 - 16) \div 3 = 18.5178$ in.



AREA OF CIRCLE.

1. $157.08 \times \frac{50}{4} = 1963\frac{1}{2}$ square inches.

2. $.7854 \times 18^2 = 254.4696$ square inches.

3. $.07958 \times 90^2 = 644.598$ square rods.

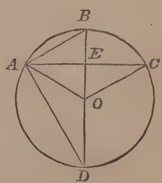
AREA OF SECTOR.

1. $18°$ is $\frac{1}{20}$ of $360°$; hence, the sector is $\frac{1}{20}$ of the circle. Area of circle $= 9 \times 3.1416$; area of sector $= \frac{1}{20}(9 \times 3.1416) = 1.41372$ square feet.

2. Turning to figure under the problems for finding the length of an arc, we see that the sector subtended by any arc is half as great as that subtended by twice the arc; hence, the sector corresponding to the given arc will be half the required sector. Let AB be the given arc, $\overline{AC}^2 - \overline{AE}^2 = \overline{EC}^2$, or $2500 - 225 = 2275$; $\sqrt{2275} = 47.696$. Then, ED = radius $- EC = 2.304$. $\overline{ED}^2 + \overline{AE}^2 = \overline{AD}^2$, or $5.308416 + 225 = 230.308416$; whence, $AD = 15.1759$, the chord of $\frac{1}{2}$ the given arc. According to Art. 16, $(15.1759 \times 8 - 30) \div 3 = 30.469$, length of given arc; $30.469 \times 25 = 761.725$, area of sector subtended by given arc. Doubling this, we have 1523.45, area of the required sector.

AREA OF SEGMENT.

1. Let ABC be the segment, AC its chord and BE its altitude, and draw the diameter BD perpendicular to the chord. Then, $\overline{AB}^2 = \overline{AE}^2 + \overline{EB}^2 = 104$. Drawing AD, the triangle ABD is right-angled; hence (Bk. III., Th. XIV., 2), $\overline{AB}^2 = BD \times BE$;



$\therefore BD = \frac{\overline{AB}^2}{BE} = \frac{104}{2} = 52$, diameter; $\sqrt{104} = 10.198$, chord of $\frac{1}{2}$ the arc; $\frac{10.198 \times 8 - 20}{3} = 20.528 =$ length of arc; $20.528 \times 13 = 266.864$, area of sector $ABCO$; $BO - EB = EO = 24$, altitude of triangle AOC; $24 \times 10 = 240$, area of triangle; $266.864 - 240 = 26.864$, area of segment.

2. Taking the figure of the previous example, we have $EO = BO - BE = 7$. Then, in the triangle AEO, $\overline{AE}^2 = \overline{AO}^2 - \overline{EO}^2 = 576$; $\therefore AE = 24$. In the triangle ABE, $\overline{AB}^2 = \overline{AE}^2 + \overline{BE}^2 = 900$; $\therefore AB = 30$, chord of $\frac{1}{2}$ the arc. $AC = 48$, chord of arc. $\frac{30 \times 8 - 48}{3} = 64$, length of arc; $64 \times \frac{25}{2} = 800$, area of sector. $\frac{AC \times EO}{2} = 168$, area of triangle AOC. $800 - 168 = 632$ square inches.

3. An arc of $180°$ is $\frac{1}{2}$ circumference; hence, the segment is a semicircle, $\frac{3.1416 \times 12^2}{2} = 226.1952$, area of semicircle.

AREA OF CIRCULAR RING.

1. $225 - 100 = 125$; $3.1416 \times 125 = 392.70$.

2. $200^2 - 176^2 = 9024$; $3.1416 \times 9024 = 28349.7984$ sq. feet $= 3149.9776$ square yards.

SIDE OF AN INSCRIBED SQUARE.

1. $14 \times .7071 = 9.8994$ inches.

2. $400 \times .2251 = 90.04$ inches.

THE ELLIPSE.

1. $10 \times 8 \times 3.1416 = 251.328$.

2. $3 \times \frac{5}{2} \times 3.1416 = 23.562$ sq. feet.

MENSURATION OF VOLUMES.

THE PRISM.

CONVEX SURFACE.

1. $(6+7+8) \times 50 = 1050$ square inches.

2. $16 \times 16 \times 6 = 1536$ square inches $= 10\frac{2}{3}$ square feet.

3. $\frac{6+7+8}{2} = 10.5$, half sum of sides; $(10.5-6)(10.5-7)(10.5-8) \times 10.5 = 413.4375$; $\sqrt{413.4375} = 20.333$, base of prism; $20.333 \times 2 = 40.666$, sum of bases; $1050 + 40.666 = 1090.666$ square inches.

CONTENTS.

1. $30 \times 30 \times 30 = 27000$ cubic inches $= 15.625$ cubic feet.

2. $4 \times 4 \times 27 = 432$ cubic feet.

3. $\frac{3+4+5}{2} = 6$; $(6-3)(6-4)(6-5) \times 6 = 36$; $\sqrt{36} = 6$, area of base; $6 \times 24 = 144$ cubic feet.

THE PYRAMID.

CONVEX SURFACE.

1. $(3+4+5)\frac{20}{2} = 120$ square feet. **2.** $5 \times 5 \times \frac{60}{2} = 750$ square feet.

CONTENTS.

1. $2.5980762 \times 5^2 \times \frac{20}{3} = 433.0127$ cubic feet.

2. $(763.4)^2 \times \frac{480}{3} = 93244729.6$ cubic feet.

THE CYLINDER.

1. $6 \times 3.1416 \times 12 = 226.1952$ square feet.

2. $3.1416 \times 8 \times 20 = 502.656$ square feet.

3. $3.1416 \times (3\frac{1}{3})^2 \times 12 = 418.88$ cubic feet.

4. Area of base $= \frac{2150.42}{8} = 268.8025$; square of radius $= \frac{268.8025}{3.1416} = 85.562293$; radius $= \sqrt{85.562293} = 9.249$; diameter $= 9.249 \times 2 = 18.498$.

THE CONE.

1. Slant height $= \sqrt{3^2+4^2} = 5$; convex surface $= 3.1416 \times 6 \times \frac{5}{2} = 47.124$ square feet. Contents $= 3.1416 \times 9 \times \frac{4}{3} = 37.6992$ cubic feet.

2. $3.1416 \times 20 \times \frac{26}{2} = 816.816 =$ convex surface. Altitude $= \sqrt{26^2 - 10^2} = 24$; contents $= 3.1416 \times 100 \times 8 = 2513.28$.

FRUSTUM OF A PYRAMID AND CONE.

CONVEX SURFACE.

1. $(4 \times 12 + 4 \times 8)\frac{24}{2} = 960$ square feet.

2. $(12+8) \times 3.1416 \times \frac{20}{2} = 628.32$ square feet.

CONTENTS.

1. Volume $= \frac{3.1416}{3}(6^2 + 3^2 + 6 \times 3) \times 40 = 2638.944$ cubic feet.

2. Area of greater base $= 2.5980762 \times 9$; area of lesser $= 2.5980762 \times 4$; square root of product $= \sqrt{2.5980762^2 \times 9 \times 4} = 2.5980762 \times 6$, contents $= (2.5980762 \times 19)\frac{24}{3} = 394.9075$ cubic feet.

3. Volume $= \frac{3.1416}{3}(15^2 + 10^2 + 15 \times 10) \times 42 = 20891.64$ cubic inches; $20891.64 \div 231 = 90.44$ gallons.

THE SPHERE.

SURFACE OF A SPHERE.

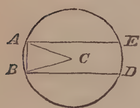
1. $3.1416 \times 17 \times 17 = 907.9224$ square inches $= 6.305$ square feet.

2. $3.1416 \times 7912^2 = 196{,}663{,}355.75$.

SURFACE OF ZONE.

1. $3.1416 \times 25 \times 6 = 471.24$ square feet.

2. Let $ABED$ represent the section of the torrid zone; then, the arc AB, which measures the height of the zone on the surface of the sphere, measures about 47°; hence, the angle ACB, which is measured by it, is an angle of 47°. Now, in the isosceles triangle ABC, the angles at A and B are equal and measure $\frac{1}{2}(180° - 47°)$, or $66\frac{1}{2}°$, each. Then, by Theorem I.,



sin BAC : sin ACB : : side BC : side AB.

sin BAC, $66\frac{1}{2}°$, a. c.	0.037602
sin ACB, 47°,	9.864127
log. side BC, 3956,	3.597256
log. side AB,	3.498985

Then, $AB \times 7912 \times 3.1416 =$ surface of zone.

log. AB,	3.498985
log. 7912,	3.898286
log. 3.1416,	0.497151
log. surface 78,419,272,	7.894422

CONTENTS OF A SPHERE.

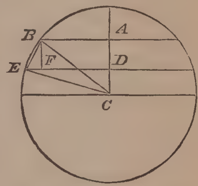
1. $17^3 \times \frac{3.1416}{6} = 2572.4468$ cubic inches.

2. $4500^3 \times \frac{3.1416}{6} = 47713050000$.

CONTENTS OF A SPHERICAL SEGMENT.

1. $(4^2+3\times 8^2)\times 4\times .5236=435.6352$ cubic inches.

2. This example is solved by the formula given Bk. VII., Th. X., Cor. 4, subtracting the second cone, as the segment is on one side of the centre. Before applying the formula, however, it is necessary to find the surface of the zone BE, the radii of the two bases of the segment, and the lines AC and DC, the altitudes of the two cones formed on the bases by the revolution of the triangles DCE and ACB.



In the isosceles triangle BCE, the angle $BCE=43°$, as we know from geography; hence, EBC and BEC are each $68\frac{1}{2}°$. Also, $BCA=23\frac{1}{2}°$; $ABC=90°-23\frac{1}{2}°=66\frac{1}{2}°$. Drawing BF perpendicular to ED, $CBF=23\frac{1}{2}°$, being alternate angle to BCA; $EBF=EBC-CBF=45°$; $BEF=90-45°=45°$; $ECD=BCE+BCA=66\frac{1}{2}°$. From triangle BCE we have the proportion,

sin BEC, a. c.	0.031322
: sin BCE,	9.833783
: : side BC,	3.597256
: side BE,	3.462361

and from triangle BEF,

rad., a. c.	0.000000
: sin BEF,	9.849485
: : side BE,	3.462361
: side BF,	3.311846

Now, multiplying surface of zone by $\frac{1}{3}$ of radius, we have

Altitude zone, BF,	3.311846
diameter of circle, 7912,	3.898286
π	0.497151
radius,	3.597256
a. c. of 3,	9.522879
volume of sector, 67207538461,	10.827418

From triangle ACB we have the proportions,

rad. a. c.,	0.000000
: sin BCA,	9.600700
: : side BC,	3.597256
: side AB,	3.197956

rad., a. c.	0.000000
: sin ABC,	9.962398
: : side BC,	3.597256
: side AC,	3.559654

We can now obtain the second term of the formula,

π,	0.497151
$\overline{AB}^2$,	6.395912
AC,	3.559654
a. c. of 3,	9.522879
volume of first cone, 9453565217,	9.975596

The triangles ACB and DCE have angle ABC = angle ECD, and angle BAC = angle EDC, and $EC = BC$; hence, $ED = AC$ and $DC = AB$.

We can now find the third term,

π,	0.497151
$\overline{ED}^2$,	7.119308
DC,	3.197956
a. c. of 3,	9.522879
volume of 2d cone, 21741700000,	10.337294

Volume temperate zone = 67207538461 + 9453565217 − 21741700000 = 54,919,403,678 cubic miles.

CYLINDRICAL RINGS.

1. $(18+4) \times 4 \times 9.8696 = 868.5248$.

2. $(2+12) \times 1 \times 9.8696 = 138.1744$ cubic inches.

SIDE OF AN INSCRIBED CUBE.

1. $16 \times .57736 = 9.23776$.

2. $(18.849552 \div 3.141592) \times .57736 = 3.46416$, side of cube; volume $= 3.46416^3 = 41.571305$ cubic inches.

VOLUME OF AN IRREGULAR BODY.

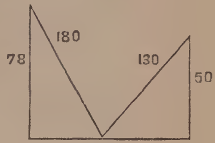
1. $3.1416 \times 25 \times 5 = 392.70$ cubic inches.

2. $2\frac{1}{2}$ quarts $= \frac{231}{4} \times 2\frac{1}{2} = 144.375$ cubic inches $= .08355$ cubic feet; $14 - .08355 = 13.91645$ cubic feet.

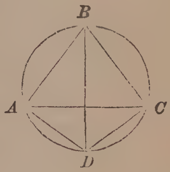
MISCELLANEOUS PROBLEMS. (p. 177.)

1. $(15.5+11.25)\times 2\times 7.75=414.625$ square feet; 30 inches $=2.5$ feet; $414.625\div 2.5=165.85$ feet $=55.2833$ yards.

2. The ladder is evidently the hypothenuse of two right-angled triangles, the sum of whose bases is the width of the street, $\sqrt{130^2-78^2}=104$; $\sqrt{130^2-50^2}=120$; $120+104=224$ feet.



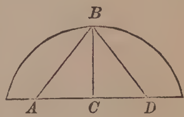
3. From any vertex, as B, of the triangle ABC, draw a diameter bisecting the triangle, and also the side AC; draw AD and DC. Then, AD and DC, being chords of half the arc AC, are sides of an inscribed hexagon, and therefore equal to radius. Then in the right-angled triangle BCD, $BC=\sqrt{\overline{BD}^2-\overline{DC}^2}=\sqrt{16-4}=2\sqrt{3}=$ side of triangle; $\frac{1}{2}$ sum of sides $=\dfrac{3\times 2\sqrt{3}}{2}=3\sqrt{3}$; $(3\sqrt{3}-2\sqrt{3})^3\times 3\sqrt{3}=27$; $\sqrt{27}=3\sqrt{3}=$ area.



4. 16 inches $=1\frac{1}{3}$ feet; $6\div 1\frac{1}{3}=4\frac{1}{2}$ feet.

5. $3.1416\times 6\times 6\times 144\times .08=\$1302.884$.

6. Let ABD be the triangle; then, the perpendicular BC let fall upon the base will be the radius of the circle; square of side $=100\div .4330127=230.940108$; square of $\frac{1}{2}$ side $=\dfrac{230.940108}{4}$; radius $=\sqrt{230.940108-\dfrac{230.940108}{4}}=13.16074$; diameter $=26.32148$.



7. The area of the semicircle is 100 feet, and of the circle 200 feet. Then, square of diameter $=200\div .7854=254.64731347+$; diameter $=15.9576$ feet.

8. Area of garden $=80\times 60=4800$; area of walk $=\dfrac{4800}{2}=2400$, and perimeter of garden $=(80+60)\times 2=280$. Let $x=$ width of walk; then, $280x-4x^2=2400$; whence, $x=10$ feet.

9. Area of circle $=1000\times\frac{1}{4}(1000\div 3.1416)=79577.2854$. Dividing this area by 11.1961524, we have $7107.5564+$, square of one side; hence, one side $=84.306$, and perimeter $=84.306\times 12=1011.672$ feet.

10. Circles are to each other as the squares of their radii; hence, 1 acre : $\frac{49}{9}$ acres : : $178^2 : x^2$; $x^2 = \frac{178^2 \times 49}{9}$; hence, $x = \frac{178 \times 7}{3} = 415\frac{1}{3}$ links = 4 chains $15\frac{1}{3}$ links.

11. Let x = diameter of B's; then, $1 : 6\frac{1}{4} :: 20^2 : x^2$; $x^2 = 6\frac{1}{4} \times 20^2$: hence, $x = 2\frac{1}{2} \times 20 = 50$ rods.

12. Area of A's $= \left(\frac{64}{3.1416}\right)^2 \times .7854 = 325.948$ square rods; side of B's $= \frac{64}{4} = 16$; area of B's $= 16^2 = 256$; difference $= 325.948 - 256 = 69.948$ square rods.

13. Circumference of circle $= \sqrt{\frac{640}{3.1416}} \times 2 \times 3.1416 = \sqrt{640 \times 4 \times 3.1416}$ $= 89.6799$ rods. Perimeter of square $= \sqrt{640} \times 4 = 101.1928$; $101.1928 - 89.6799 = 11.5129$ rods.

14. $\frac{1}{10}$ of an acre = 16 square rods; side of yard $= \sqrt{16} = 4$ rods; contents of remainder $= \frac{49}{64}$ of 16; side of remainder $= \sqrt{\frac{49}{64} \text{ of } 16} = 3\frac{1}{2}$ rods; $4 - 3\frac{1}{2} = \frac{1}{2}$ rod; width of walk $= \frac{1}{2}$ of $\frac{1}{2}$ a rod $= \frac{1}{4}$ of a rod = 4 feet $1\frac{1}{2}$ inches.

15. Difference of squares $= 500 - 100 - 1 = 399$; difference of sides = 3; sum of sides $= 399 \div 3 = 133$; side of lesser square $= \frac{133 - 3}{2} = 65$; number of men $= 65^2 - 100 = 4125$.

VOLUMES.

1. Volume $= \frac{1}{6}\pi D^3 = 606.132$; $D^3 = \frac{606.132 \times 6}{\pi}$; $D = 10.5$ feet.

2. Diameter of a sphere equals side of cube; hence, volume $= \frac{1}{6}$ of $3.1416 \times 216 = 113.0976$ cubic feet.

3. Contents of cistern $= 3.1416 \times (36 + 20\frac{1}{4} + 27)\frac{5}{3} = 435.897$ cubic feet $= 753230.016$ cubic inches; $\frac{753230.016}{231 \times 31\frac{1}{2}} = 103.515$ barrels.

4. $(30^2 + 15^2 + 30 \times 15) \times \frac{220}{3} = 115500$ cubic feet.

5. 100 acres = 4356000 square feet; $4356000 \times 10 = 43560000$ cubic feet.

6. Since the side of a square is to its diagonal as 1 to $\sqrt{2}$, the side of the larger end is $\frac{3}{\sqrt{2}}$ and of the smaller $\frac{2}{\sqrt{2}} = \sqrt{2}$; then, $(\frac{9}{2} + 2 + 3)\frac{20}{3}$ $= 63\frac{1}{3}$ cubic feet.

7. Let x = diameter and y = depth; then, $x^3 : (18\frac{1}{2})^3 :: 64 : 1$; $x = 74$ inches; $y^3 : 8^3 :: 64 : 1$; $y = 32$ inches.

8. Let x = the time; then, $x : 1 :: 20^3 : 5^3$; $x = 64$ days.

9. Area of garden $= 100^2 = 10000$ square feet; area remaining $= \frac{10000}{2} = 5000$ square feet; side of remainder $= \sqrt{5000} = 70.7106$; width of walk $= 100 - 70.7106 - 29.2894$ feet.

10. Quantity to be dug from ditch $= 80 \times 100 \times 1 = 8000$ cubic feet; length of ditch $= 2 \times 80 + 2 \times 100 + 4 \times 4 = 376$ feet; surface of ditch $= 376 \times 4 = 1504$ square feet; $8000 \div 1504 = 5.319$ feet.

11. 1 cubic foot = 1728 cubic inches; area of one end of wire $= 3.1416 \times \left(\frac{1}{60}\right)^2$; $1728 \div \frac{3.1416}{3600} = 1728 \times \frac{300}{.2618} = 1980137.509$ in. $= 31.252$ miles.

12. Since volume and surface are numerically equal, $\frac{4}{3}\tau R^3 = 4\pi R^2$; whence, $\frac{1}{3}R = 1$, or $R = 3$ and $D = 6$.

13. Contents of whole shell $= \frac{1}{6}$ of $3.1416 \times 4^3 = 33.5104$; contents of hollow part $= \frac{1}{6}$ of $3.1416 \times 2^3 = 4.1888$; difference $= 33.5104 - 4.1888 = 29.3216$ cubic inches of metal; $29.3216 \times \frac{1}{4} = 7.3304$ pounds.

14. From problem 4, the whole contents are found to be 115500 cubic feet; $\frac{3.1416}{3}[(7\frac{1}{2})^2 + (5\frac{1}{2})^2 + 7\frac{1}{2} \times 5\frac{1}{2}] \times 220 = 29431.556$ cubic feet; $115500 - 29431.556 = 86068.444$ cubic feet.

15. Cubic feet to be dug from ditch $= 1 \times 300 \times 200 = 60000$; length of ditch $= 600 + 400 + 32 = 1032$ feet; surface of ditch $= 1032 \times 8 = 8256$ square feet; $60000 \div 8256 = 7$ feet 3.209 inches.

16. Area of garden $= 80 \times 100 = 8000$ square feet; area of walk $= \frac{8000}{2} = 4000$ square feet.

Then, let x = one side of remaining area, and $x + 20$ = the other; $x^2 + 20x = 4000$; whence, $x = 54.0312$; $80 - 54.0312 = 25.9688$ feet.

17. Let x and y represent the altitudes of the pyramids formed by the sections; then, $x^3 : 20^3 :: 1 : 3$; whence, $x = 13.867$, first one's share; $y^3 : 20^3 :: 2 : 3$; whence, $y = 17.471$; second one's share $= 17.471 - 13.867 = 3.604$; third one's share $= 20 - 17.471 = 2.529$.

PLANE TRIGONOMETRY.

SOLUTION OF TRIANGLES.

CASE I. (p. 24.)

2. Sum of A and $C = 93° 25'$; $B = 180° - 93° 25' = 86° 35'$.

$\sin C\ (65° 45')$	a. c.	0.040118	
$: \sin B\ (86° 35')$		9.999227	
$:: AB\ (625)$		2.795880	
$: AC$		2.835225	$\therefore AC = 684.266.$

$\sin C\ (65° 45')$	a. c.	0.040118	
$: \sin A\ (27° 40')$		9.666824	
$:: AB\ (625)$		2.795880	
$: BC$		2.502822	$\therefore BC = 318.29.$

CASE II. (p. 24.)

2.

$BC\ (62.50)$	a. c.	8.204120	
$: AB\ (45.96)$		1.662380	
$:: \sin A\ (79° 21')$		9.992454	
$: \sin C$		9.858954	$\therefore C = 46° 16' 38''.$

$A + C = 125° 37' 38''$. $\therefore B = 180° - 125° 37' 38'' = 54° 22' 22''$.

$\sin C\ (46° 16' 38'')$	a. c.	0.141046	
$: \sin B\ (54° 22' 22'')$		9.909996	
$:: AB\ (45.96)$		1.662380	
$: AC$		1.713422	$\therefore A = 51.69.$

3.

$BC\ (15.71)$	a. c.	8.803824
$: AC\ (21.12)$		1.324694
$:: \sin A\ (27° 50')$		9.669225
$: \sin B$		9.797743

$\therefore B = 38° 52' 47''$, or $141° 7' 13''$; and $C = 113° 17' 13''$, or $11° 2' 47''$.

sin A (27° 50′) a. c.	0.330775	
: sin C (11° 2′ 47″)	9.282404	
: : BC (15.71)	1.196176	
: AB	0.809355	$\therefore AB = 6.447$.

CASE III. (p. 27.)

2. $180° - 68°\ 36' = 111°\ 24' = A + B$; half sum $= 55°\ 42'$. $240 + 360 = 600 =$ sum of sides; $360 - 240 = 120 =$ difference of sides;

600 a. c.	7.221849	
: 120	2.079181	
: : tan 55° 42′	10.166118	
: tan. ½ difference angles,	9.467148	½ dif. $= 16°\ 20'\ 26''$.

$55°\ 42' + 16°\ 20'\ 26'' = 72°\ 02'\ 26'' =$ one angle; $55°\ 42' - 16°\ 20'\ 26'' = 39°\ 21'\ 34''$.

sin 72° 02′ 26″ a. c.	0.021694	
: sin 68° 36′	9.968976	
: : 360	2.556303	
: remaining side,	2.546973	side $= 352.349$.

CASE IV. (p. 28.)

2. Let $BC = 1005$, $AC = 1210$, $AB = 1368$; then (Th. III.),

$$AB : AC + BC :: AC - BC : AD - BD;$$

or

$$1368 : 2215 :: 205 : AD - BD;$$

hence, $AD - BD = 331.926$; $AD = 849.963$, and $BD = 518.037$.

In the triangle ACD, we have

AC (1210) a. c.	6.917215	
: AD (849.963)	2.929400	
: : sin D 90°	10.000000	
: sin ACD	9.846615	$\therefore ACD = 44°\ 37'\ 26''$.

In the triangle BCD, we also have

BC a. c.	6.997834	
: BD	2.714361	
: : sin D (90°)	10.000000	
: sin BCD	9.712195	$\therefore BCD = 31°\ 1'\ 41''$.

Hence,
$$A = 90° - 44°\ 37'\ 26'' = 45°\ 22'\ 34'';$$
$$B = 90° - 31°\ 1'\ 41'' = 58°\ 58'\ 19'';$$
$$C = 44°\ 37'\ 26'' + 31°\ 1'\ 41'' = 75°\ 39'\ 7''.$$

RIGHT-ANGLED TRIANGLES. (p. 29.)

2. sin 90° a. c.	0.000000	
: sin 45° 36′	9.853986	
: : hypothenuse (45.36)	1.656673	
: perpendicular	1.510659	∴ perpendicular = 32.408.

$90° - 45°\ 36' = 44°\ 24'$.

sin 90° a. c.	0.000000	
: sin 44° 24′	9.844889	
: : hypothenuse (45.36)	1.656673	
: base	1.501562	∴ base = 31.736.

3. hypothenuse (396) a. c.	7.402305	
: base (218)	2.338456	
: : sin 90°	10.000000	
: sin angle at vertex	9.740761	∴ angle = 33° 24′ 05″.

$90° - 33°\ 24'\ 05'' = 56°\ 35'\ 55''$.

sin 90° a. c.	0.000000	
: sin 56° 35′ 55″	9.921600	
: : hypothenuse	2.597695	
: perpendicular	2.519295	∴ perpendicular = 330.59.

4. AC 58.75 a. c.	8.230992	
: AB 74.58	1.872622	
: : rad.	10.000000	
: tan C	10.103614	∴ $C = 51°\ 46'\ 15''$.

$90° - 51°\ 46'\ 15'' = 38°\ 13'\ 45''$; $\sqrt{58.75^2 + 74.58^2} = 94.94$.

PRACTICAL APPLICATIONS. (p. 30.)

Case I.

1. Angle $ACB = 90° - 48°\ 36' = 41°\ 24'$.

sin ACB a. c.	0.179594	
: sin BAC	9.875126	
: : AB	2.079181	
: BC	2.133901	∴ $BC = 136.113$ feet.

Case II.

1. Angle $ACB = 90° - 20° = 70°$.

$\sin BAC$ a. c.	0.465948	
$: \sin ACB$	9.972986	
$:: BC$	2.204120	
$: AB$	2.643054	$\therefore AB = 439.596$.

Case III.

1. $A + B = 72° 40' + 45° 36' = 118° 16'$; $180° - 118° 16' = 61° 44' = C$.

$\sin C$ $61° 44'$ a. c.	0.055146	
$: \sin A$ $72° 40'$	9.979816	
$:: AB$	2.477121	
$: BC$	2.512083	$\therefore BC = 325.15$ yards.

Also, $\sin C$ a. c.	0.055146	
$: \sin B$	9.853986	
$:: AB$	2.477121	
$: AC$	2.386253	$\therefore AC = 243.362$ yards.

Case IV.

1. $180° - 43° 16' = 136° 44' = A + B$.

$BC + AC$ 550 a. c.	7.259637	
$: BC - AC$ 50	1.698970	
$:: \tan \frac{1}{2}(A+B)$ $68° 22'$	10.401646	
$: \tan \frac{1}{2}(A-B)$	9.360253	$\therefore A - B = 12° 54' 37''$.

$A = 68° 22' + 12° 54' 37'' = 81° 16' 37''$; $B = 68° 22' - 12° 54' 37'' = 55° 27' 23''$.

$\sin A$ a. c.	0.005053	
$: \sin C$	9.835941	
$:: BC$	2.477121	
$: AB$	2.318115	$\therefore AB = 208.025$.

Case V.

1. $DBA = 90° - EDB = 21° \ 18'$; $BDA = EDB - EDA = 26° \ 14'$.

sin DBA a. c.	0.439793	
: sin BDA	9.645450	
: : DA	2.602060	
: AB	2.687303	$\therefore AB = 486.747$.

Case VI.

1. $ACB = CBD - CAB = 23° \ 45'$.

sin ACB a. c.	0.394968	
: sin CAB	9.778119	
: : AB	2.662758	
: BC	2.835845	
rad. a. c.	0.000000	
: sin CBD	9.940196	
: : BC	2.835845	
: CD	2.776041	$\therefore CD = 597.0917$.

Case VII.

1. $CAB = CAD + BAD = 98° \ 54'$; $ABD = ABC + DBC = 113° \ 26'$; $ACB = 180 - (CAB + ABC) = 36° \ 30'$; $ADB = 180 - (BAD + ABD) = 24° \ 10'$.

sin ACB a. c.	0.225612	
: sin CAB	9.994739	
: : AB	2.602060	
: CB	2.822411	$\therefore CB = 664.372$.
sin ADB a. c.	0.387860	
: sin BAD	9.828855	
: : AB	2.602060	
: BD	2.818775	$\therefore BD = 658.832$.
$CB + BD$ a. c.	6.878373	
: $CB - BD$	0.743510	
: : tan $\frac{1}{2}(BDC + BCD)$	10.164220	
: tan $\frac{1}{2}(BDC - BCD)$	7.786103	$\therefore \frac{1}{2}$ difference $= 21'$.

sin BDC a. c.	0.081767	
: sin DBC	9.969665	
:: CB	2.822411	
: BC	2.873843	$DC = 747.9$.

Case VIII.

1. $FAC = 180° - (ACF + CFA) = 47° 20'$; $DBE = 180° - (BED + BDE = 33° 11'$; $180° - ACD = 123° 24' = CDA + CAD$.

sin FAC a. c.	0.133530	
: sin CFA	9.993703	
:: CF	2.778151	
: AC	2.905384	$\therefore AC = 804.237$.

$AC + CD$ a. c.	6.852560	
: $AC - CD$	2.310134	
:: tan $\frac{1}{2}(CDA + CAD)$	10.268859	
: tan $\frac{1}{2}(CDA - CAD)$	9.431553	$\therefore \frac{1}{2}$ dif. $= 15° 6' 57''$.

$CDA = 76° 48' 57''$; $CAD = 46° 35' 3''$; $ADB = BDC - CDA = 73° 41' 3''$.

sin CDA a. c.	0.011601	
: sin ACD	9.921607	
:: AC	2.905384	
: AD	2.838592	$\therefore AD = 689.592$.

sin DBE a. c.	0.261759	
: sin BED	9.999150	
:: DE	2.778151	
: BD	3.039060	$\therefore BD = 1094.108$.

$180 - ADB = 106° 18' 57'' = ABD + BAD$.

$BD + AD$ a. c.	6.748679	
: $BD - AD$	2.606936	
:: tan $\frac{1}{2}(BAD + ABD)$	10.125378	
: tan $\frac{1}{2}(BAD - ABD)$	9.480993	$\therefore \frac{1}{2}$ dif. $= 16° 50' 25''$.

$BAD = 69°\ 59'\ 53\frac{1}{2}''$.

sin BAD a. c.	0.027019	
: sin ADB	9.982148	
: : BD	3.039060	
: AB	0.048227	$AB = 1117.44$.

Case IX.

Let the circumference of a circle be passed through A, B and P, draw AE and BE from the point where CP cuts the circumference, and also let fall a perpendicular from C upon AB at D. Then, the angles EBA and APC are equal, being measured by the same arc, and also EAB and BPC. Then, in triangle EAB we have two angles and a side given; hence, angle $AEB = 180° - 56°\ 15' = 123°\ 45'$. Hence, we have the proportion

sin AEB a. c.	0.080154	
: sin EBA	9.744739	
: : AB	2.903090	
: AE	2.727983	$\therefore AE = 534.543$.

Then, in triangle CAB, having the three sides, we find angles according to Theorem III.

AB a. c.	7.096910	
: $AC + BC$	3.000000	
: : $AC - BC$	2.301030	
: $AD - BD$	2.397940	$\therefore AD - BD = 250$.

$AD = 525$. Then, in right-angled triangle ADC,

AC a. c.	7.221849	
: AD	2.720159	
: : rad.	10.000000	
: sin ACD	9.942008	$\therefore ACD = 61°\ 2'\ 42''$.

$CAD = 90 - ACD = 28°\ 57'\ 18''$; $CAE = CAD - EAB = 6°\ 27'\ 18''$. Then, in triangle CAE, $\frac{1}{2}$ sum of $ACE + AEC = \frac{1}{2}(180 - CAE) = 86°\ 46'\ 21''$.

$AC + AE$ a. c.	6.945180	
: $AC - AE$	1.815956	
: : tan $\frac{1}{2}(AEC + ACE)$	11.248794	
: tan $\frac{1}{2}(AEC - ACE)$	10.009930	$\therefore AEC - ACE = 45°\ 39'\ 18''$.

$ACE = 41° 7' 3''$. Then, in triangle CAP,

sin APC a. c.	0.255261
: sin ACP	9.817965
: : AC	2.778151
: PA	2.851377 $PA = 710.193$.

$CAP = 180° - 74° 52' 3'' = 105° 7' 57''$.

sin APC a. c.	0.255261
: sin CAP	9.984673
: : AC	2.778151
: PC	3.018085 $\therefore PC = 1042.522$.

In triangle BPC,

BC a. c.	7.3979400
: PC	3.0180857
: : sin BPC	9.5828400
: sin PBC	9.9988657 $\therefore PBC = 85° 51' 38''$,

or its supplement. As PBC is formed by a tangent and chord, it is measured by half the arc PAB, which is evidently more than a semicircumference; hence, it is obtuse, and the value obtained from the proportion is the supplement of the angle required. Hence, $PCB = 180 - (PBC + BPC) = 63° 21' 38''$.

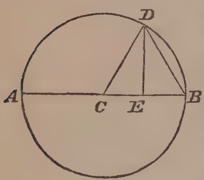
sin BPC a. c.	0.417160
: sin PCB	9.951262
: : BC	2.602060
: PB	2.970482 $\therefore PB = 934.291$.

NOTE.—The log. sin CAP will be exactly 9.9846737, but to get the value of PC, as given in the book, it is necessary entirely to disregard the last figure. In obtaining the sin PBC, however, the exact logarithm of PC must be used. These variations will make no difference practically, as they only affect the second and third decimal figures.

ANALYTICAL TRIGONOMETRY.

THEOREMS AND PROBLEMS. (p. 49.)

1. Since the chord of $60° = R$, draw radius CD and chord DB, forming the equilateral triangle DBC, of which sin 60° is the perpendicular; then, since the perpendicular of an equilateral triangle bisects the base, we have $CE = \frac{1}{2}R$, or $\frac{1}{2}CD$, or since $R = 1$, $CE = \frac{1}{2}$. Then, $DE = \sqrt{\overline{CD}^2 - \overline{CE}^2} = \sqrt{1 - \frac{1}{4}} = \frac{1}{2}\sqrt{3}$; CE $= \cos 60° \;\; \frac{1}{2}$.



2. Sin $30° = \cos 60° = \frac{1}{2}$; $\cos 30° = \sin 60°$, which by the diagram above we see equals $\frac{1}{2}\sqrt{3}$.

3. Sin 45° = cos 45°; hence, sine and cosine form two sides of a square, of which the radius is the diagonal. In fig., p. 37, $OP : ON$ $:: 1 : \sqrt{2}$; $\therefore \; OP = \dfrac{ON}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$. Or, $\overline{ON}^2 = \overline{OP}^2 + \overline{PN}^2 = \overline{2PN}^2$; hence, $PN = \dfrac{ON}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$.

4. Tan $45° = \dfrac{\sin 45°}{\cos 45°}$; $\tan 45° = \dfrac{\frac{1}{2}\sqrt{2}}{\frac{1}{2}\sqrt{2}} = 1$; but $\sec a = \dfrac{1}{\cos a}$; hence, $\sec 45° = \dfrac{1}{\frac{1}{2}\sqrt{2}} = \sqrt{2}$.

5. Sin $15° = \sin(60° - 45°)$, which equals $\sin 60° \times \cos 45° - \cos 60°$ $\times \sin 45° = \frac{1}{2}\sqrt{3} \times \frac{1}{2}\sqrt{2} - \frac{1}{2} \times \frac{1}{2}\sqrt{2} = \dfrac{(\sqrt{3}-1)\sqrt{2}}{4}$; dividing both terms by $\sqrt{2}$, we have $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$; $\cos 15° = \cos 60° \cos 45° + \sin 60° \sin 45° = \frac{1}{2} \times \frac{1}{2}\sqrt{2}$ $+ \frac{1}{2}\sqrt{3} \times \frac{1}{2}\sqrt{2} = \dfrac{(\sqrt{3}+1)\sqrt{2}}{4} = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$.

6. Tan $15° = \frac{\sin 15°}{\cos 15°} = \frac{\sqrt{3}-1}{2\sqrt{2}} \div \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$; multiplying both terms by $\sqrt{3}-1$, we have $\frac{3-2\sqrt{3}+1}{3-1} = 2-\sqrt{3}$; $\cot 15° = \frac{\cos 15°}{\sin 15°} = \frac{\sqrt{3}+1}{2\sqrt{2}} \div \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{3+2\sqrt{3}+1}{3-1} = 2+\sqrt{3}$.

7. Sin a cos $a = \frac{1}{4}\sqrt{3}$; but $\sin a = \sqrt{1-\cos^2 a}$; substituting, $\cos a\sqrt{1-\cos^2 a} = \frac{1}{4}\sqrt{3}$; squaring, $\cos^2 a - \cos^4 a = \frac{3}{16}$, which is a quadratic equation, and solving it, we have $\cos a = \frac{1}{2}$, or $\frac{1}{2}\sqrt{3}$, and $\sin a = \frac{1}{2}\sqrt{3}$ or $\frac{1}{2}$.

8. $3 \sin a + 5\sqrt{3} \times \cos a = 9$; but $\cos a = \sqrt{1-\sin^2 a}$; substituting, $3 \sin a + 5\sqrt{3} \times \sqrt{1-\sin^2 a} = 9$; transposing and squaring, $84 \sin^2 a - 54 \sin a = -6$; whence, solving the quadratic, $\sin a = \frac{1}{2}$ or $\frac{1}{7}$.

9. Sin $a(\sin a - \cos a) = \frac{4}{25}$; substituting value of $\cos a$, $\sin a(\sin a - \sqrt{1-\sin^2 a}) = \frac{4}{25}$; transposing and squaring, $2 \sin^4 a - \frac{33}{25} \sin^2 a = -\frac{16}{625}$; whence, $\sin a = \frac{4}{5}$.

10. Tan $a = \frac{\sin a}{\cos a} = \frac{\sin a}{\sqrt{1-\sin^2 a}}$; $\frac{\sin a}{\sqrt{1-\sin^2 a}} = \frac{4}{3}$; whence, $3 \sin a = 4\sqrt{1-\sin^2 a}$; squaring, $9 \sin^2 a = 16 - 16 \sin^2 a$; whence, $\sin^2 a = \frac{16}{25}$; $\sin a = \frac{4}{5}$; $\cos a = \sqrt{1-\sin^2 a} = \frac{3}{5}$.

11. Cot $a = \frac{1}{\tan a}$; substituting, $\tan a + \frac{1}{\tan a} = 2$; clearing of fractions, $\tan^2 a - 2 \tan a = -1$; whence, $\tan a = 1$.

12. Tan$^2 a - \sin^2 a = \frac{\sin^2 a}{\cos^2 a} - \sin^2 a = \frac{(1-\cos^2 a)\sin^2 a}{\cos^2 a}$; but $1-\cos^2 a = \sin^2 a$; hence, $\frac{1-\cos^2 a}{\cos^2 a} \times \sin^2 a = \frac{\sin^2 a}{\cos^2 a} \times \sin^2 a$, which equals $\tan^2 a \sin^2 a$.

13. Sec$^2 a$ cosec$^2 a = (1+\tan^2 a)(1+\cot^2 a) = 1 + \tan^2 a + \cot^2 a + \tan^2 a \cot^2 a$; but $\tan a \cot a = 1$; hence, the expression becomes $1 + \tan^2 a + 1 + \cot^2 a$, which, by substituting, equals $\sec^2 a + \operatorname{cosec}^2 a$.

14. By Formula 1, Art. 86, $\sin(30° + a) + \sin(30° - a) = 2 \sin 30° \cos a$; but by Theorem 2, $\sin 30° = \frac{1}{2}$; hence, $2 \sin 30° \cos a = \cos a$.

15. By Formula 3, Art. 86, $\cos(60° + a) + \cos(60° - a) = 2 \cos 60° \cos a$; but (Theorem 1) $\cos 60° = \frac{1}{2}$; hence, $2 \cos 60° \cos a = \cos a$.

16. Transposing, $a+b=180°-c$; whence, $\tan(a+b)=\tan(180°-c)$; or by Formula E, Art. 84, $\dfrac{\tan a+\tan b}{1-\tan a \tan b}=-\tan c$; clearing of fractions, $\tan a+\tan b=-\tan c+\tan a \tan b \tan c$; whence, $\tan a+\tan b+\tan c =\tan a \tan b \tan c$.

17. Transposing, $a+b=90-c$; whence, $\cot(a+b)=\cot(90-c)$, or by Formula G, Art. 84, $\dfrac{\cot a \cot b-1}{\cot b+\cot a}=\tan c$ or $\dfrac{1}{\cot c}$; clearing of fractions, $\cot a \cot b \cot c-\cot c=\cot b+\cot a$; whence, $\cot a+\cot b+\cot c =\cot a \cot b \cot c$.